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CONJECTURA PHYSICA

CIRCA
PROPAGATIONEM SONI
AC LUMINIS

UNA CUM
ALIIS DISSERTATIONIBUS ANALYTICIS
DE NUMERIS AMICABILIBUS
DE NATURA ÆQUATIONUM, AC
DE RECTIFICATIONE ELLIPSIS

AUCTORE
LEONHARDO EULERO.

BEROLINI,
SUMTIBUS AMBR. HAUDEVIDUË ET JOH. CAROL. SPENERI,
BIBLIOPOL. REG. ET ACAD. SCIENT. PRIVIL.
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Conjectura Physica
de
Propagatione Soni ac Luminis

§. I.

In motu corporum ac præcipue fluidorum plurima occurrunt phaenomena, quæ per Theoriam nondum explicari possunt. Et si enim principia Mechanicæ, a quibus omnes motus determinationes pendent, satis cognita atque ad quosvis casus accommodata videntur, ut eorum ope motus immutationes formulis analyticis includi queant; tamen sæpenumero ipsa Analysis his formulis evolvendis imparprehenditur. His igitur casibus non tam Mechanica, ad quam scientiam motuum investigatio proprie pertinet, imperfectionis est

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est accusanda, quam Analysis, cujus munus in resolvendis æquationibus, ad quas reliquæ Matheseos partes perduxerint, præcipue versatur. Sic quanquam Geometræ post inventum calculum infinitorum nimium studii in Analysis excolenda multis collocasse videntur; tamen frequenter a principiis mechanicis ad ejusmodi æquationes differentiales deducimur, quarum resolutio, antequam Analysis multo adhuc majora ceperit incrementa, frustra tentatur. Quamobrem eorum labor, qui omnem operam in promovendis Analyseos finibus impendunt, non solum non est reprehendendus, sed etiam quam maxime laudandus.

§. II. Hunc Analyseos defectum potissimum in Astronomia Theoretica deprehendimus. Cum enim huic scientiæ sit propositum ex viribus, quibus corpora coelestia se mutuo impellunt, eorum motus determinare, si hoc negotium minus successerit, a mechanica certe omnis suspicio culpæ removeri debet. Quæcunque enim concipiantur vires, quibus planeta quispiam sollicitetur, motus ejus semper per æquationes mære analyticas exprimi potest; ita ut accurata motus descriptio a resolutione harum æquationum, in quo Analyseos officium versatur, pendeat. At nisi virium illarum lex simplicissima statuatur, æquationes istæ tantopere fiunt implicatæ, ut omnia artificia, quæ in Analysis adhuc sunt detecta, ad eas resolvendas minime sufficiant. Atque ita Analyseos maxime defectui est tribuendum, quod motus Lunæ, cunctæque ejus inæqualitates nondum ad certas leges revocari, ac tabulis Astronomicis comprehendendi potuerint.

§. III. Interim tamen ejusmodi quoque se offerunt quæstiones ad quas expediendas non tam Analysis, quam ipsius Mechanicæ cognitio sufficiens desideratur. In solidis quidem corporibus hoc potissimum evenit, quando circa axes mobiles in gyrum aguntur: hoc enim casu ea Mechanicæ principia nobis adhuc sunt occulta, ex quibus variationem hujusmodi motuum definire liceat.

Maxime

Maxime autem iste Mechanicæ defectus in motu & agitatione fluidorum cernitur; in quo fere penitus ignoramus, qua ratione singulæ fluidi partes in se invicem agant, suosque inter se motus perturbent. Quamobrem etsi cursus fluminum, & fluxus aquarum per canales satis commode ad calculum revocari posset; tamen motum fluidorum, qui vocatur intestinus, quo singulæ fere particule inter se agitantur, nullo adhuc modo certis legibus circumscribere licuit.

§. IV. Cum igitur sonus in motu quodam vibratorio, quo minimæ aeris particule inter se commoventur, consistat; quomodo hic motus sit comparatus, & qualege ex aliis aeris particulis in alias transmittatur, accurate explicari adhuc nequit. Neque Newtonus, & qui post eum hoc negotium sunt aggressi, soni per aerem propagationem satis dilucide exposuisse sunt censendi. In dissertatione enim mea de Lumine & Coloribus, ubi Newtoni doctrinam illustravi, luculenter monstravi, Virum Acutissimum propagationem pulsuum per medium elasticum non ex solis mechanicæ principiis determinasse, sed hypothesein quampiam experimentis quidem confirmatam subtili admodum modo in subsidium vocasse; sicque primarium soni phaenomenon, quo uniformiter per aerem proferri observatur, non tam explicasse, sed potius ei investigationem suam superstruxisse, quod tamen ex sola Theoria demum deduci debuisset.

§. V. Tantum abest, ut hoc Newtoni institutum, quo cum principiis mechanicis hypotheses conjunxit, reprehendendum existimem, ut potius hanc viam ob defectum idoneorum principiorum solam esse arbitrer, quæ nos ad aliquam saltem certam cognitionem perducere valeat. Cum enim determinationem celeritatis, qua sonus per aerem promovetur, suscipit, commode & quasi præter opinionem evenit, ut hæc quantitates, quæ hypotheses introduxerant, penitus ex calculo egrediantur, atque conclusio

A 2

ab



ab istis hypothesibus ita immunis obtineatur, ut intelligi possit, etiamsi aliæ hypotheses fuissent assumptæ; eandem tamen conclusionem prodituram fuisse; quo ipso veritas conclusionis maxime confirmatur. Verum tamen etsi hoc modo celeritas, quæ pulsus per aerem aliudve fluidum elasticum transit; recte definiri videtur, tamen hinc ipsa particularum agitatio, quæ pulsus continetur, non elicitur: atque ea, quam calculus exhibet, ab hypothesibus maxime pendet, ideoque pro vera agnosci nequit.

§. VI. Porro etiam ita in celeritate pulsuum, quæ per methodum Newtoni invenitur, acquiescendum puto, ut etiamsi experientia adversari videatur, tamen inde Theoriæ nihil plane detrahatur. Invenit autem Newtonus pulsuum quemque per aerem tanta celeritate propagari oportere, ut singulis minutis secundis spatium 979 pedum Londinensium conficiat, cum tamen experimentis constet sonum singulis minutis secundis 1140 pedes percurrere. Causam quidem hujus accelerationis Newtonus in eo ponit, quod aerem plurimis ejusmodi particulis imprægnatum esse arbitratur, per quas pulsus in instanti propagentur, ita ut si aer hujusmodi particulis omnino esset repletus, sonus quoque sine mora per intervalla quantumvis magna transferretur. Quare ut ipsam soni celeritatem observatam obtineat, septimam fere aeris partem ejus naturæ assumere cogitur, ut per eam pulsus puncto temporis, quasi per corpuscula perfecte dura Cartesii, transirent.

§. VII. Verum hæc explicatio pluribus præmitur difficultatibus iisque tam gravibus, ut nullam verisimilitudinis speciem retineat. Aer enim si tanta copia corpusculorum durorum redundaret, eas proprietates, quæ ipsi inesse novimus, elasticitatem, & comprimendi facultatem penitus amitteret. Namque si septima pars, vel etiam decima, quam Newtonus assumit; hujusmodi particulis esset constata, aer certe non in minus spatium quam subseptuplum vel subdecuplum comprimi posset, ac tum etiam omnem elateris



elateris vim amittere deberet. Cum autem tam per experimenta constet, quam ex vi pulveris pyrii colligere liceat, aerem non solum in multo minus spatium redigi posse, sed etiam tum maxima vi elasticitatis gaudere, nullus amplius locus sententiæ Newtonianæ relinquitur, Præterea quoque experimentis compertum est, sonum pari celeritate per aerem transmitti, sive is majore sive minore vaporum copia sit inquinatus; unde evidens est, partium heterogenearum; quæ aeri sunt permixtæ, copiam nihil omnino ad sonum accelerandum conferre.

§. VIII. Ego vero nequidem hujusmodi subterfugio opus esse arbitror ad veritatem Theoriæ salvandam; neque enim concedendum puto, Theoriam ab experientia dissentire. Namque casus, ad quem Theoriæ est accommodata, prorsus discrepat ab eo, qui per experimenta adversari videtur. Quod quo facilius perspiciatur, recordandum est in Theoria unicum tantum pulsus, qui nullos alios habeat insequentes, considerari, cum per experimenta celeritas soni, hoc est ingentis pulsuum se mutuo insequentium frequentię declaretur. Nusquam autem in theoria probatum est, plures pulsus se mutuo quasi e vestigio insequentes eadem celeritate per medium elasticum propagari debere, quæ unus pulsus solitarius progreditur: quinpotius in loco supra allegato, ubi doctrinam Newtoni explicavi, celeritatem pulsuum ab insequentibus affici debere animadverti.

§. IX. Cum igitur nullum experimentum proferri possit, quo celeritas unici pulsus exhibeatur, propterea quod promptissimus ictus, in quo nulla mora inesse videatur, satis magnam pulsuum seriem creat: acque non solum demonstrari nequeat, pulsuum frequentiam nihil in eorum celeritate mutare, sed etiam probabile videatur, celeritatem soni, tam ob agitationem aeri jam insitam, quam ab impulsu insequentium pulsuum accelerari debere; nulla omnino pugna inter theoriam & experientiam adhuc concludi potest.

A 5

Quare



Quare potius e contrario concludere licebit, si Theoria vera sit, unicusque pulsus tantum spatium 979. pedum uno minuto secundo percurrat; majorem illam soni celeritatem, quam experientia ostendit, nulli alii causæ nisi frequentia pulsuum tribuendebere. Atque hinc generatim affirmare non dubito, quo major sit pulsuum sonum quempiam constituentium frequentia, eo celerius hunc sonum per aerem propagari.

§. X. Neque tamen hanc sententiam tanquam veritatem propono, sed tantum pro ejusmodi conjectura haberi volo, quæ fortasse, postquam plura alia phænomena consuluerimus atque ad examen vocaverimus, ad summum certitudinis gradum evehi queat. Quoniam enim ne unius quidem pulsus promotionem per aerem quietum ex solis mechanicæ principiis definire licet; multo minus hæc principia sufficient ad propagationem plurium pulsuum se invicem insequentium determinandam. Interim tamen facile intelligitur, a pulsuum frequentia eorum celeritatem non medio-criter affici debere: primo enim cum aeris particula a pulsibus antecedentibus jam sint in quapiam agitatione constitutæ, pulsus sequentes aliam inde accelerationem adipisci necesse est. Deinde quia in quolibet pulsu particulae aeris motu reciproco agitantur, et tam antroorsum quam retrorsum concitantur, necessario evenire debet, ut hæc agitatio in particulas antecedentis pulsus vim quandam exerat, quæ eo erit major, quo pulsus sibi fuerint propiores eorumque propterea frequentia major: hocque ergo casu pulsus præcedentes ab insequentibus magis propellentur, sicque soni celeritas augebitur.

§. XI. Quamvis autem hac consideratione conjectura mea jam satis probabilis videatur, tamen experimenta, quorum ingentem numerum Sollertissimus Derham omni adhibita cura instituit, contrarium nobis persuadere videntur. Compertum enim est omnis generis sonos, sive debiles, sive vehementes sive, etiam graves sive acutos

eutos pari celeritate per aerem propagari. Quod quidem ad soni vehementiam ac debilitatem attinet, ex theoria etiam colligitur, hinc in celeritate soni nullum discrimen oriri posse: verum quia soni acuti majori pulsuum frequentia constant, graves contra minori, secundum conjecturam meam soni acutiores celerius per idem spatium promoveri deberent, quam graviores; quod cum a Derhamo negetur, videndam est, an ejus experimentis vel potius conclusionibus, quas inde deduxit, tanta vis tribui queat, qua conjectura mea evertatur: neque enim sine plena eviſtione contrarii de sententia alias probabili decedere decet.

§. XII. Ac primo quidem si intervallum, quod ad experimenta instituenda deligitur, non fuerit valde magnum, nullum discrimen in velocitate sonorum acutissimorum et gravissimorum percipi poterit. Ponamus enim in spatio, quod a sonis uno minuto secundo percurritur, dari pro diversa soni indole differentiam 50 pedum, quæ autem reipsa fortasse adhuc multo minor existit. Iam si is, qui experimentum capit, a loco, ubi sonus editur, intervallo 10000 pedum sit remotus, atque sono gravissimo celeritas 1000 pedum pro minuto secundo tribuatur, acutissimo autem celeritas 1050 pedum, observator exaudiet sonum gravissimum post 10⁴, acutissimum autem semiminuto secundo tantum citius. Quod discrimen etiamsi sensibile videatur, tamen quia ipsum momentum, quo quisque sonus editur, tam accurate per signum indicari nequit, ut nullus plane error sit metuendus, merito mihi equidem dubitare videor, an experimenta Derhamiana tam sint exacta, ut ista quæstio per ea decidi possit.

§. XIII. Deinde vero non solum in observatione momenti, quo sonus editur, levis quidam error admitti potest, sed etiam in observatione ejus momenti, quo sonus primum exauditur, propterea quod satis parvas minuti secundi partes distinguere non licet, ita ut ob hanc duplicem causam error unius semiminuti

minuti secundi inevitabilis videatur. Tum vero etiam perpendendum est, nisi ambo soni, gravis et acutus, simul edantur, quod quidem institutio experimentorum vix permittit, a diversa commotione aeris quandam differentiam oriri posse; quoniam a ventis propagatio soni tam accelerari quam retardariprehenditur. Imprimis autem animadverti oportet, in distantia 10000 pedum, quam assumi, omnis generis sonos ratione gravis et acuti discrepantes, quales instrumentis musicis edi solent, exaudiri non posse. At si hujusmodi experimenta in minoribus distantis instituantur, differentia inter sonorum gravium et acutorum perceptionem adhuc multo minor evadet, omnemque observatoris diligentiam effugiet. Neque ergo conjectura, quam proposui, per experimenta ullam adhuc probabilitatis diminutionem est passa.

§. XIV. Verum objicietur, Derhamum in multo majoribus distantis etiam experimenta sua de soni velocitate instituisse, atque adeo tempus, quo tormentorum fragor per spatium 60000 pedum propagetur, esse dimensum. Sed in hujusmodi sonis, quæ a tormentis ac sclopetis eduntur, tanta non inest diversitas ratione gravis et acuti, ut inde quicquam sive ad confirmandam sive ad refellendam conjecturam meam concludi possit. Videntur autem hi soni vehementer graves; ex quo spatium 1140 pedum Anglicorum, per quod sonus singulis minutis secundis propelli ex his experimentis colligitur, sonis tantum gravissimis erit tribuendum: ita ut soni acutiores aliquanto majus spatium singulis minutis secundis percurrere sint existimandi. Quanto autem celerius soni acutiores per aerem propagentur quam graviores, nullis fere experimentis definiri posse videtur, quod cum a Theoria multo minus sit expectandum, omnino in dubio relinquitur, etsi alias ipsa conjectura jam satis probabilis videatur.

§. XV. Neque etiam hæc experimenta eum in finem commemoravi, ut inde quicquam ad conjecturam corroborandam concludi

cludi posse crederem, sed tantum ut obiectionem, quæ gravissima videbatur, diluerem, atque ostenderem ex his experimentis, quæ Derhamus summa diligentia instituit, nullum argumentum contra conjecturam meam peti posse. Quod si ergo Newtono concedendum putamus, unicum pulsus in aere non ultra 979 pedes uno minuto secundo promoveri, necesse est ut majorem illam sonorum celeritatem frequentia pulsum se invicem prosequentium et quasi propellentium tribuamus. Ex quo et illud agnoscere cogimur, quo major fuerit pulsum frequentia eo majorem quoque illam velocitatis accelerationem esse oportere, etiam si fortasse differentia in sonis gravissimis et acutissimis sit admodum exigua. Nulla enim ratio suadet, ut credamus accelerationem soni ultra spatium illud 979 pedum frequentia pulsum exacte esse proportionalem; sed fieri potest, ut cum frequentia multo sit major, tamen inde vix notabilis acceleratio oriatur.

§. XVI. Interim tamen non puto discrimen hoc in velocitate sonorum plane esse inobservabile; quantumvis enim id sit parvum, a Musicæ peritis percipi posse videtur. Quod si enim plurium diversorum sonorum concentus, in quo singuli soni tam exacte justis intervallis se invicem insequi debent, ut vel tenuissimus error aures offendat, e longinquo audiatur, discrimen facile animadvertetur, si soni acutiores unico quasi instante citius ad aures perferrentur, quam graviore. Admissa autem conjectura vox gravissima, quæ Bassus vocatur, respectu reliquarum vocum acutiorum aliquanto tardius e longinquo exaudiri deberet, quam ab adstantibus, hæcque retardatio ab auribus musicæ adfuetis multo accuratius sentietur, quam si celeritatem cujusque soni per exactissimas temporis mensuras eo, quo Derhamus usus est modo, investigare vellemus. Quod si ergo hujusmodi discrimen pro variis distantiiis, ex quibus concentus Musicus auditur, observaretur, hoc ipso conjectura nostra extra omnem dubitationem collocaretur.

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B

§. XVII.

§. XVII. Deinde si in quoque sono celeritas pulsuum antecedentibus ab insequentibus intenditur, in pulsibus ultimis hæc acceleratio nullum amplius locum habebit, iique propterea celeritate naturali propagabuntur, et singulis minutis secundis spatium tantum 979 pedum conficient, dum pulsus primi eodem tempore spatium circiter 1140 pedum absolvent. Ex quo sequitur eundem sonum, quo longius audiatur, eo diutius durare debere: ponamus enim sonum unico quasi ictu seu puncto temporis absolvi, ita ut ejus duratio adstantibus brevissima videatur; quod si jam idem sonus in distantia 10000 pedum audiatur, primus pulsus exaudietur post tempus $\frac{10000}{1140}$ //, ultimus autem pulsus demum post tempus $\frac{10000}{979}$ //, sicque totum tempus, quo hic sonus percipietur erit $\frac{10000}{979} - \frac{161}{1140}$ // seu $\frac{10000}{4912}$ //. Hinc ergo duratio soni in distantia 10000 pedum prope $1\frac{1}{2}$ sec. in distantia autem 20000 pedum tribus minutis secundis protrahetur. Difficile autem erit hanc sonorum prolongationem discernere, quia objecta interposita ob tremorem conceptum jam per se sonos protrahere solent.

§. XVIII. Multo autem fortiora argumenta ad conjecturam meam confirmandam suppeditant phænomena lucis, quæ si probe perpendantur, nullum fere amplius dubium relinquent. Cum enim pulsuum propagatio in æthere perinde lumen efficiat, ac sonus per pulsus in aere propagatos excitatur; si ostendero in æthere celeritatem pulsuum ab eorundem frequentia pendere, nullo modo dubitare licebit, quin in aere etiam pulsuum propagatio ab eorum frequentia acceleretur. Præcipuum autem lucis phænomenon, quo conjectura mea confirmari videtur, in diversa refractionis ratione versatur, quam radii diversorum colorum, dum ex alio medio diaphano in aliud transeunt, sequi observantur. Quia enim probavi colorum varietatem in nulla alia re poni posse, nisi in varia pulsuum, quibus quisque color representatur, frequen-



tia, necesse est ut diversæ refractionis rationes a diversa pulsum frequentia profisciscantur, radiique verbi gratia rubri ob eam tantum causam minorem refractionem pati sunt censendi, quam violacei, quod in illis major minorve pulsum frequentia inest, quam in his.

§. XIX. In Theoria autem mea lucis et colorum luculenter ostendi, si radius lucis ex uno medio in aliud transit, semper esse debere sinum anguli incidentiæ ad sinum anguli refractionis in eadem ratione, quam tenet celeritas pulsum in medio priori ad celeritatem eorum in medio posteriori. Quanquam autem ibi sum suspicatus frequentiam pulsum rationem refractionis immutare posse, radiosque rubros, dum ex medio rariori in densius, ubi tardius progrediuntur, transeunt, ideo minus refringi putavi, quod pulsus sequentes, quia in medio densiori propius ad se invicem accederent, refractionem diminuere videbantur; tamen hæc causa cessaret, si radii ex medio densiori in rarius ingrediantur: neque enim hoc casu ob istam causam refractionis diminui deberet, quod tamen in hoc casu æque ac in priori evenire experientia testatur. Atque hanc ob causam illam explicationem, cur radii rubri minorem semper patiantur refractionem quam cœrulei, penitus rejiciendam esse agnosco; neque jam pulsum frequentiam quicquam ad refractionem immutandam conferre posse arbitror.

§. XX. Statuo igitur, quæcunque sit pulsum frequentia, si radius lucis ex uno medio diaphano in aliud transit, eum semper ita refringi exactissime, ut sit sinus anguli incidentiæ ad sinum anguli refractionis, uti celeritas, qua pulsus in medio priori propagantur, ad eorum celeritatem in medio posteriori. Quare cum radii rubri aliam patiantur refractionem ac violacei, necesse est, ut pulsum, quibus radii rubri constituuntur, celeritas alia ratione in transitu per diversa media immutetur, atque celeritas pulsum radiorum violaceorum. Per quodvis ergo medium alia erit celeritas radio-



rum rubrorum, alia radiorum cœruleorum, et cum hi radii tantum ratione frequentiae pulsuum inter se discrepent, perspicuum est, celeritatem pulsuum simul ab eorum frequentia pendere, ita ut qui radii diversa constant pulsuum frequentia, iidem per quodvis medium diversa celeritate progrediantur.

§. XXI. Si unicus pulsus consideretur, qui a nullis insequentibus acceleretur, ejus celeritatem per quodvis medium elasticum sequenti modo determinari comperi. Concipiatur hoc fluidum vasi inclusum, ex quo ob vim elasticam per foramen in spatium omni materia vacuum erumpat, et notetur celeritas, quaecum effluet, quæ sit debita altitudini v ; qua inventa, erit celeritas, qua unicus pulsus in isto medio elastico progredietur, debita altitudini $\frac{1}{2}v$; eritque ergo hæc celeritas ad illam, qua idem fluidum in vacuum esset erupturum, ut $\sqrt{\frac{1}{2}}$ ad 1 seu ut 1 ad $\sqrt{2}$; hoc est ut latus quadrati ad suam diagonalem. Quare cum illa altitudo v sit directe ut elasticitas et inverse ut densitas medii, sequitur celeritatem, qua unus pulsus per hoc medium propagabitur, esse in ratione subduplicata composita ex directa elasticitatis et inversa densitatis. Hæcque ratio locum habebit, etiamsi ipsa pulsuum celeritas major esset minorve, quam per Theoriam Newtonianam reperitur. Quæcunque enim fere hypothesis fingatur ad pulsuum promotionem determinandam, eadem semper proportio celeritatis pro ratione densitatis et elasticitatis medii resultat.

§. XXII. Inventa autem celeritate, qua unicus pulsus per quodpiam medium propagari debet, ea celeritas, qua radius lucis per idem medium promovetur, ob pulsuum complurium successum major est existimanda. Scilicet si celeritas unicus pulsus exprimat per p , celeritas radii lucis eo major erit quam p , quo major fuerit pulsuum hunc radium constituentium frequentia. Quo hæc frequentia facilius in calculum introduci queat, sit numerus pulsuum, qui dato tempore veluti uno minuto secundo eduntur $=n$,
fre-

frequentiam hoc ipso numero n indicare licebit; seu consideretur intervallum temporis, quod inter quemlibet pulsum et proxime insequentem est interjectum, quod erit $\frac{1}{n}t$, atque frequentia pulsum erit reciproce ut hoc intervallum $1:n$, ideoque directe ut numerus n . Exponamus autem frequentiam littera x , ita ut evanescente x frequentia cesset, casusque ad unicū pulsum reducat.

§. XXIII. Proposito ergo radio lucis quocunque, cujus pulsum frequentia sit $= x$, qua color, quem hic radius repræsentat, exponitur; si iste radius per medium quoddam diaphanum progrediatur, in quo celeritas unici pulsi sit $= p$, celeritas qua ipse radius per hoc medium propagabitur, major erit quam p , atque exprimetur certa quadam functione litterarum p et x . Cujusmodi autem hæc sit functio, a priori determinare non licet: inde enim plus non liquet, quam hanc functionem ita esse comparatam, ut ea fiat $= p$, si frequentia x plane evanescat, tum vero ut crescente frequentia x , ea quoque fiat major quam p . Hujusmodi autem functiones innumerabiles imaginari licet, quæ omnes his memoratis proprietatibus sint præditæ. Sit enim X functio quæcunque ipsius x , quæ evanescat posito $x = 0$, et quæ crescente x pariter crescât, atque celeritas radii lucis hu-

jusmodi formulis exprimi poterit: $p + X$; $p(1 + X)$; $p^{\frac{1+X}{1}}$; &c. tres autem has hypotheses potissimum examinabo.

§. XXIV. Quænam autem harum formularum in natura locum habere queant, ex phænomenis refractionis propius colligere licebit; transeat enim radius lucis, cujus frequentia $= x$, in aliud medium, per quod unicus pulsum propagetur celeritate $= q$; atque in hoc medio celeritas radii erit vel $q + X$ vel $q(1 + X)$ vel $q^{\frac{1+X}{1}}$. Unde dum radius ex priori medio in hoc transit, erit sinus anguli incidentiæ ad sinum anguli refractionis, vel ut $p + X$

ad $q + X$; velut $p(1 + X)$ ad $q(1 + X)$; velut $p \frac{1 + X}{1 + X}$ ad $q \frac{1 + X}{1 + X}$.
 In secunda ergo hypothesi foret ratio refractionis ut p ad q ; idco-
 que a pulsum frequentia non penderet: cum igitur experientia
 testetur, radios qui ratione frequentiae pulsum inter se discre-
 pant, alia quoque lege refringi, hoc ipso secunda hypothesis
 evertitur.

§. XXV. Ex prima hypothesi, quæ rationem refractionis præbet $\frac{p + X}{q + X}$ sequitur, quo major fuerit frequentia pulsum x , eo magis rationem refractionis ad rationem æqualitatis accedere; si enim esset $X = \infty$, ipsa haberetur ratio æqualitatis. Cum igitur refractionis radiorum rubrorum minor sit quam radiorum violaceorum, si hæc hypothesis locum haberet, ex ea sequeretur, radios rubros majori pulsum frequentia constare, quam radios violaceos. At quia secundum tertiam hypothesin ratio refractionis $\frac{p}{1 + X} : \frac{q}{1 + X}$ eo continuo magis a ratione æqualitatis recedit, quo major fuerit pulsum frequentia x ; si ea locum haberet, sequeretur, radios rubros minori pulsum frequentia constare, quam radios violaceos. Si igitur ex aliis phænomenis pateret, utrum in radiis rubris major pulsum frequentia insit an minor, quam in radiis violaceis? simul intelligeremus, utra harum duarum hypothesium veritati magis esset consentanea.

§. XXVI. Quo autem propius ad phænomena hoc examinem accommodemus, sit x frequentia radiorum rubrorum, y frequentia radiorum violaceorum, tum vero sit Y talis functio ipsius y , qualis X est ipsius x . Deinde sit p celeritas pulsus unici, in aere, et q in vitro, quoniam quidem refractionis radiorum diversæ coloris ex aere in vitrum summa diligentia est explorata. Sit porro $1 : m$ ratio refractionis radiorum rubrorum; et $1 : n$ radiorum violaceorum ex aere in vitrum intrantium, ita ut secundum experimenta Newtoni sit $m = \frac{10}{17}$ et $n = \frac{14}{17}$. Erit

Erit ergo per primam hypothesin $\frac{p+X}{q+X} = \frac{1}{m} = \frac{77}{50}$ & $\frac{p+Y}{q+Y} = \frac{78}{50} = \frac{1}{n}$: per tertiam vero hypothesin habebitur $(\frac{p}{q})^{1+X} = \frac{1}{m}$ & $(\frac{p}{q})^{1+Y} = \frac{1}{n}$. Inde elicitur $X = \frac{mp-q}{1-m} = \frac{50p-77q}{27}$ & $Y = \frac{np-q}{1-n} = \frac{50p-78q}{28}$; hinc vero $1+X = 1 + \frac{1}{m}$: $1 + \frac{p}{q}$ & $1+Y = 1 + \frac{1}{n}$: $1 + \frac{p}{q}$.

§. XXVII. Consideretur jam aliud quodcunque medium diaphanum, per quod unicus pulsus propageetur celeritate $= r$, sitque $1 : \mu$ ratio refractionis radiorum rubrorum, et $1 : \nu$ radiorum violaceorum, qui ex aere in hoc medium ingrediuntur. Per primam ergo hypothesin erit:

$\frac{p+X}{r+X} = \frac{1}{\mu}$ & $\frac{p+Y}{r+Y} = \frac{1}{\nu}$: Ergo ob $X = \frac{mp-q}{1-m}$ & $Y = \frac{np-q}{1-n}$ habebimus $\frac{q-p}{(1-m)r+mp-q} = \frac{1}{\mu}$ & $\frac{p-q}{(1-n)r+np-q} = \frac{1}{\nu}$: unde elicimus $r = \frac{\mu p - \mu l - mp + q}{1-m} - \frac{\nu p - \nu q - np + q}{1-n}$; hincque porro æquationem

resultantem per $p-q$ dividendo: $m\nu - \mu n - m + n + \mu - \nu = 0$.

Quod si ergo detur ratio refractionis radiorum rubrorum ex aere in quodvis medium diaphanum, per hanc æquationem assignabitur ratio refractionis radiorum violaceorum.

§. XXVIII.

§. XXVIII. Quoniam novimus discrimen inter refractiones radiorum rubrorum et violaceorum esse minimum sit $1 : d$ ratio refractionis radiorum mediæ naturæ ex aere in vitrum, existente $a = \frac{31}{78}$, ac ponamus $m = a + d$ et $n = a - d$, erit $d = \frac{1}{78}$, ideoque quantitas valde parva. Deinde pro transitu radiorum ex aere in novum istud medium diaphanum sit $1 : \alpha$ ratio refractionis radiorum mediæ naturæ, ponaturque pariter $\mu = \alpha + \delta$ et $\nu = \alpha - \delta$. ob d et δ quantitates minimas hi valores pro m , n , μ et ν in æquatione ante inventa substituti dabunt:

$$ad - \alpha \delta - d + \delta = 0, \text{ ideoque } \delta = \frac{1}{1-\alpha} d$$

Si hoc medium diaphanum sit aqua, pro qua radiorum mediæ naturæ ratio refractionis statuitur $4 : 3$ seu $\alpha = \frac{3}{4}$; erit $\delta = \frac{31}{4 \cdot 11}$.

$\frac{25}{77 \cdot 78} - \frac{31}{44} d$. Sicque diversitas refractionis pro radiorum diversa natura minor est in transitu ex aere in aquam, quam ex aere in vitrum, in ratione 31 ad 44 .

§. XXIX. Sin autem tertiam hypothefin consulamus pro hac eadem refractione radiorum ex aere in novum hoc medium transeuntium, erit $(\frac{p}{r})^{1+X} = \frac{1}{\mu}$ & $(\frac{p}{r})^{1+Y} = \frac{1}{\nu}$ ideoque $1+X =$

$$\frac{1}{\frac{1}{\mu}} = \frac{1}{\frac{1}{m}} \quad \& \quad 1+Y = \frac{1}{\frac{1}{\nu}} = \frac{1}{\frac{1}{n}} \quad \text{Hinc ergo erit } 1\mu : 1\nu = \frac{1}{\frac{1}{p}} : \frac{1}{\frac{1}{q}} = \frac{1}{\frac{1}{p}} : \frac{1}{\frac{1}{q}}$$

$1m : 1n$. Quare si μ detur atque ita exprimatur ut sit $\mu = m\epsilon$ erit quoque $\nu = n\epsilon$, undè pro transitu radiorum ex aere in diversa media hæc elicitur regula, ut, quotuplicata est ratio refractionis radiorum rubrorum in unum medium respectu rationis refractionis eorum

eandem radiorum in medium alterum, multiplicata sit quoque ratio refractionis radiorum violaceorum in medium prius respectu rationis in posterius. Seu hæc conclusio ita facilius concipi potest, ut, si ratio refractionis ex mediis quovis in aliud quodeunque sit $1:\mu$ pro radiis rubris, et $1:\nu$ pro radiis violaceis, semper sit 1μ ad 1ν in eadem ratione, quomodocumque, etiam illa media ratione refractionis discrepent.

§. XXX. Ponamus ut ante $m = a + d$, $n = a - d$, et $r = a + d$ et $s = a - d$, eritque, $1(a + d) : 1(a - d) = 1(a + d) : 1(a - d)$. Vel si sit $a + d = (a - d)'$ erit quoque $a + d = (a - d)'$. Quo autem tunc facilius valorem ipsius d elicere queamus, hæc logarithmos in series convertamus, eritque

$$\frac{1a + d}{a} = \frac{dd}{2aa} + \frac{d^3}{3a^3} \&c. \quad \frac{1a - d}{a} = \frac{dd}{2aa} - \frac{d^3}{3a^3} \&c.$$

$$\frac{1a + d}{a} : \frac{1a - d}{a} = \frac{dd + \frac{d^3}{3a^2}}{dd - \frac{d^3}{3a^2}} \quad \text{Jam ob}$$

d et d quantitates minimas, erit proxime $\frac{1a + d}{a} : \frac{1a - d}{a} = \frac{a + d}{a - d}$.

Si pro medio hoc aquæ accipiatur, ut sit $a = \frac{3}{4}$ reperietur $d = \frac{977}{1225} d$, at prima hypothesis dederat $d = \frac{11}{14} d$. In fractionibus decimalibus erit secundum primam hypoth. $d = 0.7045 d$, atque secundum tertiam hypoth. $d = 0.7630 d$; discrimen quo hic valor illum excedit est $= 0.0585 d$.

§. XXXI. Perfpicitur ergo, quantumvis hæc duæ hypotheses, quarum altera radiis rubris majorem pulsum frequentiam, altera vero minorem tribuit, quam violaceis, inter se discrepent, tamen eandem fere differentiam inter refractionem radiorum rubrorum

rubrorum ac violaceorum ex utraque oriri. Discrepantia quidem satis est parva, et per experimenta difficulter decidi possidetur, utra ad veritatem propius accedat. Quod si autem experimenta tanta cura instituantur, ut variatio refractionis pro diversa radiorum natura tam ex aere in vitrum, quam ex aere in aquam exactissime inde innotescat, non solum utra nostrarum hypotheseum sit verior, intelligitur: sed si utraque a veritate non nihil discrepare deprehendatur; facile foret novam excogitare hypothesein, quæ phaenomenis refractionis perfecte satisfaciat. Quæ inventa facilius fortasse via reddetur ad theoriam, unde hæc diversa refractionis ratio explicetur, luculentius evolvendam.

§. XXXII. Quæ hætenus de diversa radiorum lucis refrangibilitate tradidi, multo latius patent, atque locum perinde habent, siue hypothesis diversæ celeritatis, quam radiis diversicoloribus in eodem medio tribui, vera sit siue falsa. Cum enim radiorum ex alio medio in aliud transeuntium refractionem non solum a diversitate mediorum pendeat, sed etiam a colore seu natura radii, litteræ p, q, r naturam mediorum, quatenus ab ea refractionem pendet, expriment, et litteræ x et y naturam radiorum rubrorum et violaceorum, quatenus ab ea refractione afficitur. Ita si p exprimat facultatem refractivam aeris, q vitri, et r aque, tum vero x contineat naturam radiorum rubrorum et y violaceorum, quæcunque deinceps his litteris quantitates designentur, certum est si radius ruber ex aere in vitrum transeat, fore sinum anguli incidentiæ ad sinum anguli refractionis, uti est functio quæpiam litterarum p et x ad functionem similem litterarum q et x .

§. XXXIII. Denotet ϕ, p, x hanc litterarum p et x functionem, qua refrangibilitas determinatur, sitque ϕ, q, x similis functio litterarum q et x ; atque ϕ, r, x litterarum r et x . Tum pari modo pro radiis violaceis sint ϕ, y , ϕ, q et ϕ, r, y similes functiones litterarum p et y , q et y , atque r et y . Jam

pro

repantia quæ
decidi posse
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de hæc di
dam.

orum lucis
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Cum enim
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a refractione
rubrorum

Ita si p
ux, cum
aceorum,
cetum
m anguli
quæpiam
x.

et x fun
r simili
Tum
iles fun

pro radiis rubris
ex aere in vitrum intrantibus
ex aere in aquam intrantibus
pro radiis violaceis
ex aere in vitrum intrantibus
ex aere in aquam intrantibus

eritque refractiones has per functiones exprimendo

$1: m = \phi.p.x : \phi.q.x$ | $1: \mu = \phi.p.x : \phi.r.x$
 $1: n = \phi.q.y : \phi.q.y$ | $1: v = \phi.q.y : \phi.r.y$ unde habemus:

$$m = \frac{\phi.q.x}{\phi.p.x}; n = \frac{\phi.q.y}{\phi.p.y}; \mu = \frac{\phi.r.x}{\phi.p.x}; v = \frac{\phi.r.y}{\phi.p.y}$$

§. XXXIV. Per experimenta autem, quibus diversa radiorum refrangibilitas ex aere in vitrum est investigata, compertum est esse $m = \frac{1}{77}$, et $n = \frac{1}{78}$ ita ut sit $m: n = 78:77$. Quanta autem pro radiis ex aere in aquam aliudve medium ingredientibus sit differentia inter litteras μ et v , ex indole functionis ϕ definiri debet, quæ etiam si sit ignota, tamen hinc ostendere licet, non esse $\mu: v = m: n$. Si enim esset $\mu: v = m: n$ foret $\phi.r.x: \phi.r.y = \phi.q.x: \phi.q.y$, ideoque foret etiam $\phi.p.x: \phi.p.y = \phi.q.x: \phi.q.y = \phi.r.x: \phi.r.y$: hinc autem sequeretur fore $\frac{\phi.q.y}{\phi.p.y} = \frac{\phi.q.x}{\phi.p.x}$ ac per consequens $n = m$. Quare cum non sit $n = m$, manifestum est fieri non posse ut, sit $m: n = \mu: v$; sicque cum sit $m: n = 78:77$, pro radiis ex aere in aquam ingredientibus, non erit $\mu: v = 78:77$; quæ ratio etiam in nullo alio transitu radiorum ex aere in aliud medium quodcumque locum habere potest.

§. XXXV. Ex data autem differentia litterarum m et n , quæ ad transitum radiorum ex aere in vitrum pertinet, definire posse videtur differentia inter litteras μ et v , quæ transitum radiorum ex aere in aquam aliudve medium diaphanum spectant, si ratio

Sin. incid. ad sin. refract.

ut $1: m$

ut $1: \mu$

ut $1: n$

ut $1: v$



tio quædam physica in subsidium vocetur. Concipiatur nempe medium quoddam A, cujus densitas quam minime superet densitatem aeris, sitque ratio refractionis radiorum rubrorum ex aere in hoc medium intrant $\equiv 1 : M$, radiorum violaceorum vero $\equiv 1 : N$. Tum concipiantur alia media diaphana A^2, A^3, A^4, A^5 , &c. quorum densitates ita ordine crescant, ut radiorum ex quolibet medio in proxime sequens eadem sit refractionis, quæ ex aere in medium primum A. Scilicet sit

Ratio refractionis	radiorum rubrorum	radiorum violaceorum
ex aere in medium A	$1 : M$	$1 : N$
ex medio A in medium A^2	$1 : M$	$1 : N$
ex medio A^2 in medium A^3	$1 : M$	$1 : N$
ex medio A^3 in medium A^4	$1 : M$	$1 : N$

&c.

§. XXXVI. His positis manifestum est, si radii ex aere immediate ingrediuntur in medium A^2 , fore rationem refractionis pro rubris $\equiv 1 : M^2$, et pro violaceis $\equiv 1 : N^2$: simili modo si radii ex aere immediate transeant in medium A^3 , erit ratio refractionis radiorum rubrorum $\equiv 1 : M^3$, et violaceorum $\equiv 1 : N^3$. Generatim ergo si medium quoddam concipiatur signo A^z respondens, in quod radii ex aere penetrent, erit ratio refractionis radiorum rubrorum $\equiv 1 : M^z$ et radiorum violaceorum $\equiv 1 : N^z$. Quoniam igitur exponens z omnes omnino numeros complectitur, medium A^z ad omnia plane diaphana media representanda erit aptum. Quare si A^z sumatur ad vitrum exhibendum, erit $M^z \equiv m$ et $N^z \equiv n$, ideoque logarithmici sumendis z $1M \equiv 1m$ et $1N \equiv 1n$, ita ut sit $1m : 1n \equiv 1M : 1N$.

§. XXXVII. Simili modo si aliud concipiatur medium A^z , quod ratione refractionis cum aqua conveniat, erit radiorum rubrorum ex aere in hoc medium ingredientium ratio refractionis $\equiv 1 :$

$\mu = 1$; M , et radiorum violaceorum $\mu = 1$; N , ita ut jam sit $\mu = M$ et $v = N$, ideoque $l\mu : lv = 1M : 1N$. Unde patet si in transitu ex aere in aquam aliudve medium diaphanum sit ratio refractionis radiorum rubrorum $\mu = 1$; μ et radiorum violaceorum $\mu = 1$; v , fore semper $l\mu : lv = 1m : 1n$, quæ est eadem proprietates, quam supra hypothesis tertia suppeditaverat; sicque hæc hypothesis præ reliquis omnibus veritati consentanea videtur. Ex quo sequitur, si v sit velocitas unici pulsus per medium quoddam diaphanum, atque ϕ exprimat pulsum, qui radium lucis constituent, frequentiam, vel functionem ejus quampiam, fore celeritatem hujus radii per istud medium $v^{1+\phi}$; simul vero hinc est concludendum in radiis rubris minimam inesse pulsum frequentiam, in violaceis vero maximam, propterea quod illi minime, hi vero maxime refringuntur.

§. XXXVIII. Si ratio refractionis ex aere in medium quoddam diaphanum C sit pro radiis rubris $\mu = 1$; μ et pro radiis violaceis $\mu = 1$; v ; tum vero aliud medium diaphanum habeatur D , in quod radiorum ex aere incidentium sit ratio refractionis pro rubris $\mu = 1$; m et pro violaceis $\mu = 1$; n , erit $l\mu : lv = 1m : 1n = 1m : 1n$. Transeant jam radii ex medio C in medium D erit ratio refractionis radiorum rubrorum $\mu = 1$; m et radiorum violaceorum $\mu = 1$; n , atque logarithmus illius rationis erit ad logarithmum hujus rationis, ut $l \frac{\mu}{m}$ ad $l \frac{\mu}{n}$ hoc est ut $l\mu - 1m$ ad $lv - 1n$, quæ ratio redit ad hanc $1m : 1n$, sicque in transitu radiorum ex medio quocunque diaphano in aliud quodcunque, semper erit logarith. rationis refractionis radiorum rubrorum ad logar. rationis refractionis violaceorum in ratione constante.

§. XXXIX. Hæc autem ratio constans, cum sit $m = \frac{50}{77}$ et $n = \frac{50}{78}$, erit $l \frac{50}{77}$ ad $l \frac{50}{78}$, et quia eorundem numerorum

logarithmi eandem inter se tenent ratio^{em}, quicumque valor sub-
tangentis logarithmicæ tribuatur, sumendis logarithmis vulgari-
bus erit hæc ratio constans $\equiv 1875207 : 1931246$ seu proxime
ut 33 ad 34. Quodsi ergo radii ex medio quocunque in aliud
medium quodcunque ingredientur, erit semper log. rationis refra-
ctionis radiorum rubrorum, ad log. rationis refractionis radio-
rum violaceorum ut 33 ad 34. Hinc si ratio refractionis radio-
rum rubrorum per experimenta fuerit explorata, quæ sit ut 1 ad
 \mathcal{M} , ex ea facile ratio refractionis radiorum violaceorum, quæ sit ut
1 ad \mathcal{N} , definietur, cum enim sit $1\overline{\mathcal{M}} : 1\overline{\mathcal{N}}$ seu $1\mathcal{M} : 1\mathcal{N} = 33 :$
34 erit $1\mathcal{N} = \frac{34}{33} 1\mathcal{M}$, ideoque $\mathcal{N} = \mathcal{M}^{\frac{34}{33}} = \mathcal{M}^{1 + \frac{1}{33}}$.

§. XL. Sit ut supra (26) x frequentia pulsuum radiorum
rubrorum, y frequentia violaceorum, et X, Y functiones illæ harum
quantitatum, quarum ratio in definienda celeritate radiorum ha-
beri debet. Sit præterea p celeritas unici pulsus in medio priori,
et q celeritas unici pulsus in posteriori, eritque ratio refractionis
radiorum rubrorum $= p^{\frac{1+X}{1+Y}} : q^{\frac{1+X}{1+Y}}$, et violaceorum $=$
 $p^{\frac{1+Y}{1+Y}} : q^{\frac{1+Y}{1+Y}}$: hinc logarithmi harum rationum inter se erunt ut
33 ad 34, seu erit $1+X : 1+Y = 33 : 34$. Si igitur daretur
numerus X , ex eo definiri posset numerus Y : vel etiam si ratio nu-
merorum X & Y innotesceret, inde uterque assignari posset. Sed
quia hic nos Theoria deserit, nihil amplius hinc concludere
licet, etiamsi hæc hypothesis veritati maxime
consentanea videatur.

De

De Numeris Amicabilibus.

Definitio.

§. I.

Bini Numeri vocantur amica- biles, si ita sint comparati, ut sum-
ma partium aliquotarum unius æqualis sit alteri numero, &
vicissim summa partium aliquotarum alterius priori numero
æquetur.

Sic isti numeri 220 & 284 sunt amica- biles; prioris enim 220
partes aliquotæ junctim sumtæ: $1 + 2 + 4 + 5 + 10 + 11 + 20$
 $+ 22 + 44 + 55 + 110$ faciunt 284: & hujus numeri 284 partes
aliquotæ: $1 + 2 + 4 + 71 + 142$ producant priorem numerum
220.

Scholion.

§. II. Scifelius, qui primus hujusmodi numerorum mentio-
nem fecit, casu hos duos numeros 220 & 284 contemplatus ad
hanc speculationem deductus videtur; analysin enim ineptam ex-
istimat, cujus ope plura istiusmodi numerorum paria inveniantur.
Cartesius vero analysin ad hoc negotium accommodare est cona-
tus, regulamque tradidit, qua tria talium numerorum paria eli-
cuit, neque præter ea Schootenius, qui multum in hac investiga-
tione desudasse videtur, plura eruere valuit. Post hæc tempora
nemo fere Geometrarum ad hanc quæstionem magis evolvendam
operam impendisse reperitur. Cum autem nullum sit dubium quin
analysis quoque ex hac parte incrementa non contemnenda sit con-
secutura, si methodus aperlatur, qua multo plura hujusmodi nu-
merorum paria investigare liceat, haud abs re fore arbitror, si me-
thodos quasdam huc spectantes, in quas forte incidi, communica-
vero. In hunc finem autem sequentia præmittere ne-
cesse est.

Hy-

Hypothesis

§. III. Si n denotet numerum quemcumque integrum positivum, cujusmodi numeri hic semper sunt intelligendi, omnium ejus divisorum summam hoc signo $\sum n$ indicabo, ita ut character \sum numero cuiusdam præfixus summam omnium ejusdem numeri divisorum denotet: sic erit $\sum 6 = 1 + 2 + 3 + 6 = 12$.

Corollarium I.

§. IV. Quoniam inter divisores cujusvis numeri hic ipse numerus refertur, partes aliquotæ autem censentur divisores, ipse numero excepto, manifestum est summam partium aliquotarum numeri n exprimi per $\sum n - n$.

Coroll. 2.

§. V. Quoniam numerus primus nullos alios divisores admittit præter unitatem & se ipsum, si n sit numerus primus, erit $\sum n = 1 + n$. Cum autem $\sum 1 = 1$, patet unitatem non recte numeris primis annumerari.

Lemma I.

§. VI. Si m & n fuerint numeri inter se primi, ut præter unitatem nullum habeant divisorem communem, tum erit $\sum mn = \sum m \cdot \sum n$, seu summa divisorum producti mn æqualis est producto ex summis divisorum utriusque numeri m & n .

Productum enim mn primo habet singulos divisores utriusque factoris m & n , tum vero insuper divisibile est per producta ex singulis divisoribus numeri m in singulos divisores numeri n . Hi verò omnes ipsius mn divisores junctim procedunt, si m per $\sum n$ multiplicetur.

Coroll. 1.

§. VII. Si numerorum m & n uterque sit primus, ideoque $\sum m = 1 + m$ & $\sum n = 1 + n$, erit summa divisorum producti $\sum mn = (1 + m)(1 + n)$.



($1+m$)($1+n$) = $1+m+n+mn$. Si præterea p sit numerus primus diversus ab m & n , erit $fmnp = fmn.sp = fm.sn.sp = (1+m)(1+n)(1+p)$ Hincque summa divisorum cujusque numeri, qui est productum ex quocunque numeris primis diversis, facile assignabitur.

Coroll. 2.

§. VIII. Si m , n , & p non quidem sint numeri primi, sed tamen ejusmodi, ut præter unitatem nullum habeant divisorem communem, tum mn & p erunt numeri inter se primi, ac propterea $fmp = fm.sp$. Cum autem sit $fmp = fm.sn$: erit $fmp = fm.sn.sp$.

Scholion.

§. IX. Nisi factores m , n , p sint numeri inter se primi, summa divisorum producti, prout per lemma indicatur, non est justa. Cum enim secundum lemma singuli divisores factorum m , n , p inter divisores producti mnp referantur, si haberent divisorem communem, is inter divisores producti bis numeraretur; at dum questio de summa divisorum cujuspiam numeri instituitur, nullum divisorem bis numerare oportet. Hinc si m & n sint numeri primi ac $m = n$, non erit $fnp = fm.sn = (1+n)^2 = 1+2n+nn$, sed habebitur $fnp = 1+n+nn$, neque divisorem n bis poni convenit. Cum igitur per hoc lemma summa divisorum cujusque numeri, qui est productum ex quocunque numeris primis diversis, recte assignetur, residuum est, ut pro factoribus æqualibus regula tradatur, cujus ope summa divisorum producti definiri queat.

Lemma. 2.

§. X. Si n sit numerus primus, erit $fn^2 = 1+n+n^2$, $fn^3 = 1+n+n^2+n^3$; $fn^4 = 1+n+n^2+n^3+n^4$, & generatim erit

$$fn^k = 1+n+n^2+\dots+n^k = \frac{n^{k+1}-1}{n-1}.$$

Euleri Opuscula Tom. II.

Co-



Coroll. 1.

§. XI. Cum sit $fn = 1 + n$, erit $fn^2 = fn + n^2$, vel etiam $fn^2 = 1 + nfn$. Simili modo erit $fn^3 = fn^2 + n^3$, vel etiam $fn^3 = 1 + nfn^2$; porroque $fn^4 = fn^3 + n^4$ seu $fn^4 = 1 + fn^3$ & ita porro. Sicque ex cognita summa divisorum cujusque potestatis n facile summa divisorum potestatis sequentis n^{k+1} assignatur, cum sit $fn^{k+1} = fn^k + n^{k+1}$ seu $fn^{k+1} = 1 + nfn^k$.

Coroll. 2.

§. XII. Quo summæ divisorum facilius per factores expressi queant, notandum est esse $fn^2 = (1+n)(1+n^2) = (1+n^2)fn$; $fn^5 = (1+n^2+n^4)fn$; $fn^7 = (1+n^2+n^4+n^6)fn = (1+n^4)(1+n^3)fn$: sicque summæ divisorum potestatum imparium semper per factores exhiberi possunt: at potestatum parium summæ divisorum quandoque erunt numeri primi.

Coroll. 3.

§. XIII. Hinc igitur facile tabula condi poterit, qua non solum numerorum primorum, sed etiam potestatum ipsorum summæ divisorum exhibeantur. Cujusmodi Tabulam hic adjicere visum est, in qua omnium numerorum primorum millenario non majorum, eorumque potestatum ad tertiam usque & altiores pro minoribus summæ divisorum per factores expressæ traduntur.

Num.

Nam.	Summa Divisorum.	Numeri	Summ. Divisorum.	Numeri	Summa Divisorum.
2	3	3	2 ¹	11	2 ¹ . 3
2 ¹	7	3 ¹	13	11 ¹	7. 19
2 ²	3. 5	3 ²	2 ¹ . 5	11 ²	2 ¹ . 3. 61
2 ³	31	3 ³	11 ¹	11 ³	5. 3221
2 ⁴	3 ¹ . 7	3 ⁴	2 ¹ . 7. 13	11 ⁴	2 ¹ . 3 ¹ . 7. 19. 37
2 ⁵	127	3 ⁵	1093	11 ⁵	43. 45319
2 ⁶	3. 5. 17.	3 ⁶	2 ¹ . 5. 41	11 ⁶	2 ¹ . 3. 61. 7321
2 ⁷	7. 73	3 ⁷	13. 757	11 ⁷	719. 1772893
2 ⁸	3. 11. 31	3 ⁸	2 ¹ . 11 ¹ . 61	11 ⁸	2 ¹ . 3. 5. 3221. 13421
2 ⁹	2 ¹ . 89	3 ⁹	23. 3851		
2 ¹⁰	3 ¹ . 5. 7. 13	3 ¹⁰	2 ¹ . 5. 7. 13. 73	13	2. 7
2 ¹¹	8191	3 ¹¹	797161	13 ¹	3. 61
2 ¹²	3. 43. 127	3 ¹²	2 ¹ . 547. 1093	13 ²	2 ¹ . 5. 7. 17
2 ¹³	7. 31. 151	3 ¹³	11 ¹ . 13. 4561	13 ³	30941
2 ¹⁴	3. 5. 17. 257	3 ¹⁴	2 ¹ . 5. 17. 41. 193	13 ⁴	2. 3. 7. 61. 157
2 ¹⁵	131071			13 ⁵	5229043
2 ¹⁶	3 ¹ . 7. 19. 73	5	2. 3	13 ⁶	2 ¹ . 5. 7. 17. 14281
2 ¹⁷	524287	5 ¹	31		
2 ¹⁸	3. 5 ¹ . 11. 31. 41	5 ²	2 ¹ . 3. 13	17	2. 3 ¹
2 ¹⁹	7 ¹ . 127. 337	5 ³	11. 71	17 ¹	307
2 ²⁰	3. 23. 89. 683	5 ⁴	2. 3 ¹ . 7. 31	17 ²	2 ¹ . 3 ¹ . 5. 29
2 ²¹	47. 178481	5 ⁵	19531	17 ³	88741
2 ²²	3 ¹ . 5. 7. 13. 17. 241	5 ⁶	2 ¹ . 3. 13. 313	17 ⁴	2. 3 ¹ . 7. 13. 307
2 ²³	31. 601. 1501	5 ⁷	19. 31. 829		
2 ²⁴	3. 2731. 8191	5 ⁸	2. 3. 11. 71. 521	19	2 ¹ . 5
2 ²⁵	7. 73. 262657			19 ¹	3. 127
2 ²⁶	3. 5. 29. 43. 113. 127	7	2 ¹	19 ²	2 ¹ . 5. 181
2 ²⁷	233. 1103. 2089	7 ¹	3. 19	19 ³	151. 911
2 ²⁸	3 ¹ . 7. 11. 31. 151. 331	7 ²	2 ¹ . 5 ¹	19 ⁴	2 ¹ . 3. 5. 7 ¹ . 127
2 ²⁹	2147483647	7 ³	2801		
2 ³⁰	3. 5. 17. 257. 65537	7 ⁴	2 ¹ . 3. 19. 43	23	2 ¹ . 3
2 ³¹	7. 23. 89. 595479	7 ⁵	29. 4733	23 ¹	7. 79.
2 ³²	3. 43691. 131071	7 ⁶	2 ¹ . 5 ¹ . 1201	23 ²	2 ¹ . 3. 5. 53
2 ³³	3 ¹ . 71. 127. 122921	7 ⁷	3 ¹ . 19. 37. 1063	23 ³	292561
2 ³⁴	3 ¹ . 5. 7. 13. 19. 37. 73. 109	7 ⁸	2 ¹ . 11. 191. 2801		
2 ³⁵	223. 616318177.	7 ⁹	329554457.		
		7 ¹⁰			

D 2

Num.



Num.	Sum Divisorum	Num.	Sum Divisorum	Num.	Sum Divisorum	Num.	Sum Divisorum
29	2.3.5	67	2 ¹ .17	109	2.5.11.	163	2 ¹ .41
29 ¹	13.67	67 ¹	3.7 ¹ .31	109 ¹	3.7.571	163 ¹	3.7.19.67
29 ²	2 ¹ .3.5.421	67 ²	2 ¹ .5.17.449	109 ²	2 ¹ .5.11.13.457	163 ²	2 ¹ .5.41.2657
31	2 ¹	71	2 ¹ .3 ¹	113	2.3.19	167	2 ¹ .3.7
31 ¹	3.331	71 ¹	5113	113 ¹	13.991	167 ¹	28057
31 ²	2 ¹ .13.37	71 ²	2 ¹ .3 ¹ .2521	113 ²	2 ¹ .3.5.19.1277	167 ²	2 ¹ .3.5.7.2789
37	2.19	73	2.37	127	2 ¹	173	2.3.29
37 ¹	3.7.67	73 ¹	3.1801	127 ¹	3.5419	173 ¹	67.449
37 ²	2 ¹ .5.2603	73 ²	2 ¹ .5.13.57.41	127 ²	2 ¹ .5.1613	173 ²	2 ¹ .3.5.29.41.73
41	2.3.7	83	2 ¹ .3.7	131	2 ¹ .3.11	179	2 ¹ .3 ¹ .5
41 ¹	1723	83 ¹	19.367	131 ¹	17293	179 ¹	7.4603
41 ²	2 ¹ .3.7.29.	83 ²	2 ¹ .3.5.7.13.53	131 ²	2 ¹ .3.11.8581	179 ²	2 ¹ .3 ¹ .5.37.433
43	2 ¹ .11	89	2.3 ¹ .5	137	2.3.23	181	2.7.13
43 ¹	3.631	89 ¹	8011	137 ¹	7.37.73	181 ¹	3.79.139
43 ²	2 ¹ .5 ¹ .11.37	89 ²	2 ¹ .3 ¹ .5.17.233	137 ²	2 ¹ .3.5.23.1877	181 ²	2 ¹ .7.13.16381
47	2 ¹ .3	97	2.7 ¹	139	2 ¹ .5.7	191	2 ¹ .3
47 ¹	37.61	97 ¹	3.3169	139 ¹	3.13.499	191 ¹	7.13 ¹ .31
47 ²	2 ¹ .3.5.13.17	97 ²	2 ¹ .5.7 ¹ .941	139 ²	2 ¹ .5.7.9661	191 ²	2 ¹ .3.17.29.37
53	2.3 ¹	101	2.3.17	149	2.3.5 ¹	193	2.97
53 ¹	7.409	101 ¹	10303	149 ¹	7.31.103	193 ¹	3.7.1783
53 ²	2 ¹ .3 ¹ .5.281	101 ²	2 ¹ .3.17.5101	149 ²	2 ¹ .3.5 ¹ .11.101	193 ²	2 ¹ .5 ¹ .97.149
59	2 ¹ .3.5	103	2 ¹ .13	151	2 ¹ .19	197	2.3 ¹ .11
59 ¹	3541	103 ¹	3.3571	151 ¹	3.7.1093	197 ¹	19.2053
59 ²	2 ¹ .3.5.1741	103 ²	2 ¹ .5.13.1061	151 ²	2 ¹ .13.19.877	197 ²	2 ¹ .3 ¹ .5.11.3881
61	2.31	107	2 ¹ .3 ¹	157	2.79	199	2 ¹ .5 ¹
61 ¹	3.13.97	107 ¹	7.13.127	157 ¹	3.8269	199 ¹	3.13267
61 ²	2 ¹ .3.1.1861	107 ²	2 ¹ .3 ¹ .5 ¹ .229	157 ²	2 ¹ .5 ¹ .17.2579	199 ²	2 ¹ .5 ¹ .19801

Num.



Divisorum.

211 ¹	2 ⁵ .53	263 ¹	2 ⁵ .3.11	313 ¹	2.157	373 ¹	2.11.17
211 ²	3.13.31.37	263 ²	7 ⁴ .13.109	313 ²	3.181 ²	373 ²	3.7 ⁴ .13.73
211 ³	2 ⁵ .53.113.197	263 ³	2 ⁵ .3.5.11.6917.	313 ³	2 ⁵ .5.97.101.157.	373 ³	2 ⁵ .5.11.17.13913
223 ¹	1.7	269 ¹	2.3 ⁵ .5	317 ¹	2.3.53	379 ¹	2 ⁵ .5.19
223 ²	3.16651	269 ²	13.37.151	317 ²	7.14401	379 ²	3.61.787
223 ³	2 ⁵ .5.7.4973.	269 ³	2 ⁵ .3 ⁵ .5.97.373.	317 ³	2 ⁵ .3.5.13.53.773	379 ³	2 ⁵ .5.19.71821
227 ¹	2 ⁵ .3.19	271 ¹	2 ⁵ .17	331 ¹	2 ⁵ .83	383 ¹	2 ⁵ .3
227 ²	73.709	271 ²	3.24571	331 ²	3.7.5233	383 ²	147073
227 ³	2 ⁵ .3.5.19.5153.	271 ³	2 ⁵ .17.36721	331 ³	2 ⁵ .29.83.1889.	383 ³	2 ⁵ .3.5.14669
229 ¹	2.5.23	277 ¹	2.139	337 ¹	2.13 ²	389 ¹	2.3.5.13
229 ²	3.97.181	277 ²	3.7.19.193	337 ²	3.43.883	389 ²	7.21673
229 ³	2 ⁵ .5.13.23.2017	277 ³	2 ⁵ .5.139.7673	337 ³	2 ⁵ .5.13 ² .41.277	389 ³	2 ⁵ .3.5.13.29.2609
233 ¹	2.3 ⁵ .13	281 ¹	2.3.47	347 ¹	2 ⁵ .3.29	397 ¹	2.199
233 ²	7.7789	281 ²	109.727	347 ²	7.13.1327	397 ²	3.31.1699
233 ³	2 ⁵ .3 ⁵ .5.13.61.89	281 ³	2 ⁵ .3.13.47.3037	347 ³	2 ⁵ .3.5.29.12041	397 ³	2 ⁵ .5.199.15761
239 ¹	2 ⁵ .3.5	283 ¹	2 ⁵ .71	349 ¹	2.5 ⁵ .7	401 ¹	2.3.67
239 ²	19.3019	283 ²	3.73.367	349 ²	7.19.2143	401 ²	7.23029
239 ³	2 ⁵ .3.5.13 ⁴	283 ³	2 ⁵ .5.71.8009	349 ³	2 ⁵ .5 ⁵ .7.60901	401 ³	2 ⁵ .3.37.41.53.67
241 ¹	2.11 ²	293 ¹	2.3.7 ²	353 ¹	2.3.59	409 ¹	2.5.41
241 ²	3.19441	293 ²	86143.	353 ²	19.6577	409 ²	3.51857
241 ³	2 ⁵ .11 ² .113.257	293 ³	2 ⁵ .3.5 ² .7 ² .17.101	353 ³	2 ⁵ .3.5.17.59.733	409 ³	2 ⁵ .5.41.83648
251 ¹	2 ⁵ .3 ⁵ .7	307 ¹	2 ⁵ .7.11	359 ¹	2 ⁵ .3 ⁵ .5	419 ¹	2 ⁵ .3.5.7
251 ²	43.1471	307 ²	3.43.733	359 ²	7.37.499	419 ²	13.13537
251 ³	2 ⁵ .3 ⁵ .7.17 ² .109	307 ³	2 ⁵ .5 ⁵ .7.11.13.29	359 ³	2 ⁵ .3 ⁵ .5.13.4957	419 ³	2 ⁵ .3.5.7.41.2141
257 ¹	2.3.43	311 ¹	2 ⁵ .3.13	367 ¹	2 ⁵ .23	421 ¹	2.211
257 ²	61.1087	311 ²	19.5107	367 ²	3.13.3463	421 ²	3.59221
257 ³	2 ⁵ .3.5 ⁵ .43.1321	311 ³	2 ⁵ .3.13.137.353	367 ³	2 ⁵ .5.23.13469	421 ³	2 ⁵ .13.17.214.401

D 3

Num.

Num.



Num.	Summ. Divisorum	Num.	Summa Divisor.	Num.	Summa Divisorum	Num.	Summa Divisorum
431	2 ⁴ . 3 ³	479	2 ³ . 5	547	2 ¹ . 137	601	2. 7. 43
431 ²	7. 67. 357	479	43. 5347	547 ²	3. 163. 613	601 ²	3. 13. 9277
431 ³	2 ⁴ . 3 ³ . 293. 317	479	2 ⁶ . 3. 5. 89. 1285	547 ³	2 ¹ . 5. 137. 25921	601 ³	2. 7. 43. 313. 577
433	2. 7. 3 ¹	487	2 ¹ . 61	557	2. 3 ¹ . 31	607	2 ¹ . 19
433 ²	3. 37. 1693	487 ²	3. 7. 11317	557 ²	7 ¹ . 6343	607 ²	3. 13. 9463
433 ³	2 ² . 5. 7. 31. 18749	487 ³	2 ⁴ . 5. 37. 61. 641	557 ³	2 ¹ . 3 ¹ . 5 ¹ . 17. 31. 73.	607 ³	2 ¹ . 5 ¹ . 19. 73. 69
439	2 ¹ . 5. 11	491	2 ² . 3. 41	563	2 ² . 3. 47	613	2. 3. 47
439 ²	3. 51 ² . 67	491 ²	37. 6529	563 ²	31. 10243	613 ²	3. 125461
439 ³	2 ⁴ . 5. 11. 173. 557	491 ³	2 ³ . 3. 41. 145. 809	563 ³	2 ³ . 5. 29. 47. 1093	613 ³	2 ² . 5. 53. 307. 709
443	2 ¹ . 3. 37	499	2 ² . 5 ¹	569	2. 3. 5. 19	617	4. 3. 103
443 ²	7. 28099	499 ²	3. 7. 105 ²	569 ²	7 ² . 6619	617 ²	57. 3911
443 ³	2 ¹ . 3. 5 ² . 37. 157	499 ³	2 ¹ . 5 ¹ . 13. 61. 157	569 ³	2 ² . 3. 5. 19. 161881	617 ³	2 ¹ . 3. 5. 103. 38069
449	2. 3 ² . 5 ²	503	2 ¹ . 3 ² . 7	571	2 ² . 11. 13	619	2 ¹ . 5. 31
449 ²	97. 2083	503 ²	13. 19501	571 ²	3. 7. 103. 151	619 ²	3. 19. 6733
449 ³	2 ² . 3 ² . 5 ² . 100801	503 ³	2 ⁴ . 3 ² . 5. 7. 25301	571 ³	2 ² . 11. 13. 163041	619 ³	2 ¹ . 5. 13. 31. 14737
457	2. 229	509	2. 3. 5. 17	577	2. 17 ²	631	2 ¹ . 79
457 ²	3. 7. 9967	509 ²	43. 6037	577 ²	3. 19. 5851	631 ²	3. 307. 433
457 ³	2 ¹ . 5 ¹ . 229. 4177	509 ³	2 ² . 3. 5. 17. 281. 461	577 ³	2 ¹ . 5. 15 ² . 17 ² . 197	631 ³	2. 79. 159081
461	3. 3. 7. 11	521	2. 3 ² . 29	587	2 ² . 3. 7 ²	641	2. 3. 107
461 ²	173. 571	521 ²	31 ² . 283	587 ²	547. 631	641 ²	7. 58789
461 ³	2 ¹ . 3. 7. 11106261	521 ³	2 ² . 3 ² . 29. 135721	587 ³	2 ² . 3. 5. 7 ² . 34457	641 ³	2 ¹ . 3. 107. 205441
463	2 ⁴ . 29	523	2 ² . 131	593	2. 3 ² . 11	643	2 ² . 7. 23
463 ²	3. 19. 3769	523 ²	3. 13. 7027	593 ²	163. 2161	643 ²	3. 97. 1423
463 ³	2 ² . 5. 13. 17. 29. 57	523 ³	2 ¹ . 5. 7. 131. 1609	593 ³	2 ² . 3 ¹ . 5 ² . 11. 13541	643 ³	2 ¹ . 5 ¹ . 7. 23. 8269
467	2 ² . 3 ² . 13	541	4. 71	599	2 ¹ . 3. 5 ²	647	2 ¹ . 3 ⁴
467 ²	19. 11503	541 ²	3. 7. 13963	599 ²	7. 51343	647 ²	211. 1987
467 ³	2 ¹ . 3 ¹ . 5 ² . 13. 193	541 ³	2 ² . 13. 271. 11257	599 ³	2 ² . 3. 5 ² . 17. 61. 173	647 ³	2 ⁴ . 5 ⁴ . 5. 41. 1021

Num.



Summa Divisorum	Num.	Summa Divisorum	Num.	Summa Divisorum	Num.	Summa Divisorum	Num.
1.7.43	653	2.3.109	719	2 ⁴ .3 ² .5	773	2.3 ² .43	839
1.13.9277	653 ²	7.13 ² .19 ³	719 ²	487.1063	773 ²	598303	839 ²
2.7.43.3577	653 ³	2 ² .3 ² .5.109.42641	719 ³	2 ² .3 ² .5.53.4877	773 ³	2 ² .3 ² .5.43.59753	839 ³
1.19	653	2 ² .3.5.11	727	2 ² .7.13	787	2 ² .197	853
1.13.5463	659 ²	13.33457	727 ²	3.176419	787 ²	3.37 ² .151	853 ²
1.5.197369	659 ³	2 ² .3.5.11.17.53.241	727 ³	2 ⁴ .5.7.13.73109	787 ³	2 ² .5.197.241.257	853 ³
1.347	661	2.331	733	2.367	797	2.3.7.19	857
1.125461	661 ²	3.145861	733 ²	3.19.9439	797 ²	157.4051	857 ²
1.5.53.307769	661 ³	2 ² .331.218461	733 ³	2 ² .5.13.367.4133	797 ³	2 ² .5.7.19.63521	857 ³
1.3.103	673	2.337	739	2 ² .5.37	809	2.3 ⁴ .5	859
1.7.3911	673 ²	3.151201	739 ²	3.7.26041	809 ²	7.13.19.379	859 ²
1.3.5.103.3869	673 ³	2 ² .5.337.45293	739 ³	2 ² .5.37.273061	809 ³	2 ² .3 ² .5.229.1429	859 ³
1.5.31	677	2.3.113	743	2 ² .3.31	811	2.29	863
1.9.733	677 ²	459007	743 ²	552793	811 ²	3.31.73.97	863 ²
1.5.13.31.1471	677 ³	2 ² .3.5.113.45833	743 ³	2 ² .3.5 ² .31.61.181	811 ³	2 ² .7.13.29.4.67	863 ³
1.79	683	2 ² .3 ² .19	751	2 ⁴ .47	821	2.3.137	877
1.307.433	683 ²	7.66739	751 ²	3.7.26893	821 ²	7.229.421	877 ²
1.79.159081	683 ³	2 ² .3 ² .5.19.46649	751 ³	2 ² .47.282001	821 ³	2 ² .3.137.337021	877 ³
1.107	691	2 ² .173	757	2.379	823	2 ² .103	881
1.58749	691 ²	3.19.8389	757 ²	3.13.14713	823 ²	3.7.43.751	881 ²
1.107.205441	691 ³	2 ² .173.193.1237	757 ³	2 ² .5 ² .73.157.379	823 ³	2 ² .5.103.67733	881 ³
1.23	701	2.3 ² .13	761	2.3.127	827	2 ² .3 ² .23	883
1.97.1423	701 ²	492103	761 ²	579883	827 ²	684757	883 ²
1.7.23.8269	701 ³	2 ² .3.13.17.97.149	761 ³	2 ² .3.17.27.17033	827 ³	2 ² .3 ² .5.13.23.5261	883 ³
1.34	709	2.5.71	769	2.5.7.11	829	2.5.83	887
1.1987	709 ²	1.7.21971	769 ²	3.31.6367	829 ²	3.211.1087	887 ²
1.5.41.1021	709 ³	2 ² .5.37.71.6793	769 ³	2 ² .5.7.11.71.17393	829 ³	2 ² .5.7.2.29.41.53	887 ³

Num.

Num.	Summa Divisorum	Num.	Summa Divisorum.
907	2 ³ . 227	971	2 ³ . 3 ⁵
907 ²	3. 7. 39217	971 ²	13. 79. 919
907 ³	21. 5 ² . 227. 16453	971 ³	21. 3 ⁵ . 197. 2393
911	2 ⁴ . 3. 19	977	2. 3. 163
911 ²	830833	977 ²	7. 136501
911 ³	2 ⁵ . 3. 19. 39. 41. 349	977 ³	2 ⁵ . 3. 5. 53. 163. 1801
919	2 ³ . 5. 23	983	2 ³ . 3. 41
919 ²	3. 7. 13. 19. 163	983 ²	103. 9391
919 ³	2 ⁴ . 5. 23. 37. 101. 113	983 ³	2 ⁴ . 3. 5. 13. 41. 7433
929	2. 3. 5. 31	991	2 ⁵ . 31
929 ²	157. 5503	991 ²	3. 7. 13 ² . 277
929 ³	2 ³ . 3. 5. 31. 431521	991 ³	2 ⁵ . 31. 491041
937	2. 7. 67	997	2. 499
937 ²	3. 292969	997 ²	3. 13. 31. 823
937 ³	2 ² . 5. 7. 67. 87797	997 ³	2 ² . 5. 499. 99401
941	2. 3. 157		
941 ²	811. 1093		
941 ³	2 ³ . 3. 13. 157. 34057		
947	2 ² . 3. 79		
947 ²	7. 277. 463		
947 ³	2 ³ . 3. 5. 79. 89681		
953	2. 3 ² . 53		
953 ²	181. 5023		
953 ³	2 ² . 3 ² . 5. 53. 90821		
967	2 ³ . 11 ²		
967 ²	3. 67. 4657		
967 ³	2 ⁴ . 5. 11 ² . 13. 7193		

Scho-

Scholion.

§. XIV. *Uſus hujus tabulæ eſt ampliſſimus in quæſtionibus circa diviſores & partes aliquotas verſantibus reſolvendis. Ejus enim ope cujuſque numeri propoſiti ſumma diviſorum facili negotio inveniri poteſt, qua reperta, ſi inde ipſe numerus propoſitus auferatur, remanebit ejus ſumma partium aliquotarum. Ex quo ſtatim conſtat, hujus tabulæ ſubſidio numeros amicabiles, quos ſum traditurus, facile explorari poſſe, utrum ſint juſti nec ne? Quemadmodum autem ope hujus tabulæ cujuſvis numeri ſumma diviſorum cognoſci poſſit, in ſequenti lemma explicabo.*

Lemma. 3.

§. XV. *Propoſito quocunque numero ejus ſumma diviſorum ſequentimodo colligitur.*

Cum omnis numerus ſit vel primus vel productum ex primis, reſolvatur numerus propoſitus in ſuos factores primos, & qui inter ſe fuerint æquales, conjunctim exprimantur. Hoc modo numerus propoſitus ſemper ad hujusmodi formam redigetur

$m^a \cdot n^c \cdot p^y \cdot q^d$ &c. exiſtentibus m, n, p, q , &c. numeris primis. Poſito

ergo numero propoſito $\equiv N$ cum ſit $N, \equiv m^a \cdot n^c \cdot p^y \cdot q^d$ &c. & facto-

res m^a, n^c, p^y, q^d &c. inter ſe primi; erit $fN \equiv fm^a \cdot fn^c \cdot fp^y \cdot fq^d$ &c.

& valores fm^a, fn^c, fp^y, fq^d &c. ex tabula adjuncta patebunt.

1. Exmpl. Sit numerus propoſitus $N \equiv 360$.

Reſoluto hoc numero in ſuos factores primos erit $N \equiv 2^3 \cdot 3^2 \cdot 5$, ideoque $f360 \equiv f2^3 \cdot f3^2 \cdot f5 \equiv 3 \cdot 5 \cdot 13 \cdot 2 \cdot 3$, ob $f2^3 \equiv 3 \cdot 5; f3^2 \equiv 13; f5 \equiv 2 \cdot 3$.

Unde his factoribus ordinatis fiet $f360 \equiv 2 \cdot 3^2 \cdot 5 \cdot 13 \equiv 1170$.

Euleri Opuscula Tom. II.

E

2. Exem-



2. Exmpl. Explorentur numeri 2620 & 2924 utrum sint am-
icabiles nec ne?

Cum sit $2620 = 2^3 \cdot 5 \cdot 131$ & $2924 = 2^2 \cdot 17 \cdot 43$, examen
ita instituetur.

Numeri propositi	2620	2924
per factores expressi	$2^3 \cdot 5 \cdot 131$	$2^2 \cdot 17 \cdot 43$
summæ divisorum	7. 6. 132	7. 18. 44
sive	5544	5544
Summæ partium aliquotarum	2924	2620

Cum igitur summæ partium aliquotarum sint numeris reci-
proce æquales, patet propositos numeros esse amicabiles.

Scholion.

§. XVI. His igitur præmissis, quæ ad inventionem divi-
sorum cujusque numeri pertinent, ipsum problema de investiga-
tione numerorum amicabilium aggrediar, atque scrutabor, quem-
admodum hujusmodi numeros ratione summæ divisorum inter se
comparatos esse oporteat, quo deinceps facilius eorum inventio per
regulas post tradendas suscipi queat.

Problema generale.

§. XVII. Invenire numeros amicabiles, hoc est duos numeros
hujus indolis, ut alter æqualis sit summæ partium aliquotarum alterius.

Solutio.

Sint m & n duo hujusmodi numeri amicabiles, & per hy-
pothesin σm & σn summæ divisorum eorundem. Erit numeri m
summa partium aliquotarum $= \sigma m - m$, & numeri n summa parti-
um aliquotarum $= \sigma n - n$. Hinc ex natura numerorum amicabili-
um nascuntur hæc duæ æquationes:

fin



$$fm - m = n \text{ \& \& } fn - n = m$$

$$\text{five } fm = fn = m + n.$$

Numeri ergo amica**bi**les m & n primo habere debent eandem summam divisorum, tum vero oportet, ut hæc communis divisorum summa æqualis sit aggregato ipsorum numerorum $m + n$.

Coroll. 1.

§. XVIII. Problema ergo huc redu**ci**tur, ut quærantur duo ejusmodi numeri, qui habeant eandem divisorum summam, hæcque æqualis sit aggregato ipsorum numerorum.

Coroll. 2.

§. XIX. Ipsa quidem problematis ratio exigit, ut bini numeri quæ**si**ti sint inter se inæ**qua**les: sin autem desiderentur æ**qua**les, ut sit $m = n$, fiet $fm = 2n$ & $fn - n = n$: hujus scilicet numeri geminati n summa partium aliquotarum ipsi fiet æ**qua**lis, quæ est proprietas numeri perfecti. Ergo quilibet numerus perfectus repetitus numeros exhibet amica**bi**les.

Coroll. 3.

§. XX. Sin autem numeri amica**bi**les m & n , ut natura quæ**sti**onis postulat, sint inæ**qua**les, manifestum est, alterum esse redundantem alterum deficientem; summa scilicet partium aliquotarum alterius ipso erit major, alterius vero ipso minor.

Scholion.

§. XXI. Ex hac quidem generali proprietate parum adjumenti consequimur ad numeros amica**bi**les inveniendos, eo quod ista analyseos species, cujus ope æ**qua**tionem $fm = fn = m + n$ evolvere liceat, etiamnunc penitus sit inculta. Ob quem defectum formulas magis particulares contemplari cogimur, ex quarum indole regulas speciales pro inventione numerorum amica**bi**lium de-

E 2

rivare



rivare liceat; quorsum etiam pertinet regula Cartesiana a Schotenio commemorata. Ac primo quidem, etiamsi non confiet, utrum dentur numeri amicabile inter se primi nec ne? formulas generales ita restringam, ut numeri amicabiles factorem communem obtineant.

Problema Particulare.

§. XXII. *Invenire indolem numerorum amicabilem, qui communem habeant factorem.*

Solutio.

Sit a communis factor numerorum amicabilem, quorum alter ponatur $= am$, alter $= an$; sint vero tam m & a , quam n & a numeri inter se primi, ut utriusque divisorum summa per præcepta data reperiri queat. Cum igitur primo utriusque eadem esse debeat divisorum summa, fiet $fa. fm = f. fn$, ideoque $fm = fn$. Deinde vero necesse est ut sit $fu. fm$ seu $fa. fn$ ipsorum numerorum æqualis aggregato $am + an$, unde habetur $\frac{a}{fa} = \frac{fm}{m+n} = \frac{fn}{m+n}$

Positis ergo numeris amicabilibus am & an , primo esse oportet $fm = fn$, tum vero requiritur ut sit $a(m+n) = fa. fm$.

Coroll. 1.

§. XXIII. Si ergo pro m & n ejusmodi numeri jam fuerint eruti ut sit $fm = fn$: tum numerus a investigari debet, ut sit $\frac{a}{fa} = \frac{fm}{m+n}$, seu ex ratione, quam numerus ad summam divisorum suorum tenere debet, ipse numerus a erit investigandus.

Coroll. 2.

§. XXIV. Si factor communis a fuerit datus, quæstio ad inventionem numerorum m & n reducitur, qui prouti vel primi vel



na a Schote-
onflet, utrum
formulas ge-
a communem

vel compositi ex duobus pluribusve primis assumuntur, quoniam
tum divisorum summæ actū exhiberi possunt, regulæ speciales ad
eos inveniendos tradi poterunt.

Coroll. 3.

um, qui com-

§. XXV. Stātim autem perspicitur utrumque numerum m
& n primum esse non posse: quare casus simplicissimus extat, si
alter primus, alter vero productum ex duobus numeris primis as-
sumatur. Tum uterque productum ex duobus, pluribusve nu-
meris primis statui poterit, unde innumeræ regulæ speciales pro
inveniendis numeris amicabilibus derivari poterunt.

Scholion.

um, quorum
a, quam n & a
ma per præ-
usque eodem
oque $\frac{fm}{fn} = \frac{fa}{fn}$
numerosum
 $\frac{fm}{fn} = \frac{fa}{fn}$

§. XXVI. Diversæ ergo numerorum amicabilium formæ,
quæ hinc nascuntur, sequenti modo repræsentari poterunt. Sit
 a utriusque communis factor, & p, q, r, s &c. numeri primi, quo-
rum nullus sit divisor communis factoris a : atque numerorum
amicabilium formæ erunt:

$\frac{m+n}{m+n}$
esse oportet
 $\frac{m}{n}$.

Forma Prima - - - $\left\{ \begin{array}{l} apq \\ ar \end{array} \right.$

Forma Secunda - - - $\left\{ \begin{array}{l} apq \\ ars \end{array} \right.$

neri jam sue-
ber, ut sit $\frac{a}{fa}$

Forma Tertia - - - $\left\{ \begin{array}{l} apqr \\ as \end{array} \right.$

nam diviso-
gandus.

Forma Quarta - - - $\left\{ \begin{array}{l} apqr \\ ast \end{array} \right.$

Forma Quinta - - - $\left\{ \begin{array}{l} apqr \\ astu \end{array} \right.$

&c.

questio ad
i vel primi
vel

Quoniam numerus harum formarum in infinitum augeri po-
test,



test, minime tamen hinc concludere licet, in his formis omnes numeros amicales contineri. Primum enim, dum hic litteræ p, q, r, s, t , &c. numeros primos diversos significant, non verisimile est, nullos dari numeros amicales, in quibus non occurrant potestates ejusdem numeri primi. Deinde pariter non constat, utrum non dentur numeri amicales, qui vel nullum habeant factorem communem a , vel in quibus factor hic non prorsus sit idem:

veluti si darentur numeri amicales hujus formæ $m^p \cdot P \& m^q \cdot Q$, in quibus exponentes $a \& c$ essent diversi; quæ forma propterea in superioribus non contineretur, etiam si $P \& Q$ essent producta ex meris numeris primis inter se diversis. Ex his perspicitur quæstionem de numeris amicabilibus latissime patere; eamque ob hoc ipsum tam esse difficilem, ut solutio completa vix sit expectanda. Solutionibus igitur particularibus equidem tantum incumbam, & varias methodos aperiam, quarum ope ex formulis traditis plures numeros amicales mihi elicere licuit. Quælibet autem forma duplicem mihi suspexit methodum, prout factor communis a vel datus assumitur, vel ipse quæritur; hasque methodos in sequentibus problematibus exponam.

Problema. I.

§. XXVII. Invenire numeros amicales primæ formæ $a \cdot p \cdot q \& ar$, si factor communis a sit datus.

Solutio.

Cum p, q , & r sint numeri primi, atque $sr = sp \cdot sq$ seu $r + 1 = (p + 1)(q + 1)$, ponatur $p + 1 = x$ & $q + 1 = y$, fietque $r = xy - 1$. Ideoque x & y ejusmodi esse oportet numeros, ut tam $x - 1$, & $y - 1$ quam $xy - 1$ sint numeri primi. Deinde ut $a(x - 1)(y - 1)$ & $a(xy - 1)$ sint numeri amicales, oportet ut eorum aggregatum $a(2xy - x - y)$ æquale sit summæ divisorum

rum alterutrius xy fa : unde nanciscimur hanc æquationem xy fa
 $= 2axy - ax - ay$ seu $y = \frac{ax}{(2a-fa)x-a}$. Sit brevitatis gratia
 $\frac{a}{2a-fa} = \frac{b}{c}$, & $\frac{b}{c}$ sit valor fractionis $\frac{a}{2a-fa}$ ad minimos
 terminos reductæ, eritque $y = \frac{bx}{cx-b}$ seu $cy = \frac{bex}{cx-b} =$

$b + \frac{bb}{cx-b}$, unde habebimus $(cx-b)(cy-b) = bb$. Cum
 igitur $cx-b$ & $cy-b$ sint factores ipsius bb , quadratum cogni-
 tum bb in ejusmodi binos factores resolvi debet, quorum uterque
 numero b auctus fiat per c divisibilis, & quoti x & y inde emer-
 gentes ita sint comparati, ut $x-1$, $y-1$, & $xy-1$ evadant
 numeri primi. Quæ conditio quoties obtineat poterit, quodqui-
 dem pro quovis valore ipsius a assumpto statim dispicitur, toties
 obtinebuntur numeri amiables, qui erunt $a(x-1)(y-1)$ &
 $a(xy-1)$ Q. E. J.

Coroll.

§. XXVIII. Prout igitur pro a alii alique numeri accipi-
 untur, unde valores b & c innotescant, regulæ emergent parti-
 culares, quarum ope numeri amiables, si qui in eo genere dan-
 tur, facile eruentur.

Regula. I.

§. XXIX. Sit factor communis a potestas quæcunque bi-
 narij, puta $a = 2^n$ erit $fa = 2^{n+1} - 1$, ideoque $2a - fa = 1$,
 unde erit $\frac{a}{2a-fa} = 2^n$, & propterea $b = 2^n$ & $c = 1$. Hinc
 oritur $(x-2^n)(y-2^n) = 2^{2n}$.

Quare

Quare cum 2^{2n} alios non habeat factores nisi potestates bina-
rii, erit:

$$\begin{array}{l} x-2 = 2^{\frac{n}{n}} = 2^{\frac{n+k}{n-k}} \text{ seu } x = 2^{\frac{n+k}{n-k}} + 2^{\frac{n}{n}} \\ y-2 = 2^{\frac{n}{n}} = 2^{\frac{n+k}{n-k}} + 2^{\frac{n}{n}} \end{array}$$

Quocirca dispendium est, an ejusmodi valor pro k detur, ut
sequentes tres numeri

$$\begin{array}{l} x-1 = 2^{\frac{n+k}{n-k}} + 2 - 1 \\ y-1 = 2^{\frac{n+k}{n-k}} + 2 - 1 \\ xy-1 = 2^{2n+1} + 2^{2n+k} + 2^{2n-k} - 1 \end{array}$$

fiant numeri primi. Quod si succedat erunt numeri amicabile:

$$\begin{array}{l} 2 \left(2^{\frac{n}{n}} + 2^{\frac{n+k}{n-k}} - 1 \right) \left(2^{\frac{n-k}{n-k}} + 2^{\frac{n}{n}} - 1 \right) \\ 2 \left(2^{2n+1} + 2^{2n+k} + 2^{2n-k} - 1 \right) \end{array}$$

Vel sit $n-k = m$ seu $n = m+k$, fietque

$$\begin{array}{l} x-1 = 2^{\frac{m}{2k}} \left(2^{\frac{2k}{k}} + 2^{\frac{k}{k}} \right) - 1 = q \\ y-1 = 2^{\frac{m}{2m}} \left(1 + 2^{\frac{k}{2k+1}} \right) - 1 = p \\ xy-1 = 2^{\frac{2}{2m}} \left(2^{\frac{2k}{2k+1}} + 2^{\frac{2}{3k}} + 2^{\frac{k}{k}} \right) - 1 = r \end{array}$$

qui numeri, quoties fuerint primi, præbunt numeros amicales.

Casus. I.

§. XXX. Sit $k = 1$; & numeri amicales obtinebuntur, quo-
ties sequentes tres numeri fuerint primi:

$3.2 - 1; 6.2 - 1; \& 18.2 - 1$
 Tum enim positis:

$p = 3.2 - 1; q = 6.2 - 1; \& r = 18.2 - 1$
 numeri amiables erunt: $2pq \& 2r$, ob $n = m + k = m + 1$. Hæcque est regula Cartesii a Schotenio tradita.

Exemplum. 1.

§. XXXI. Sit $m = 1$; eritque
 $p = 3.2 - 1 = 5$ numerus primus.
 $q = 6.2 - 1 = 11$ numerus primus.
 $r = 18.2 - 1 = 35$ numerus primus.
 hinc ergo oriuntur numeri amiables:

$2^3.5.11 \& 2^3.71$

Sive 220 & 284, qui sunt minimi omnium, qui exhiberi possunt.

Exempl. 2.

§. XXXII. Sit $m = 2$, eritque $2 = 4 \& 2 = 16$ atque
 $p = 3.4 - 1 = 11$ numerus primus.
 $q = 6.4 - 1 = 23$ numerus primus.
 $r = 18.16 - 1 = 287$ numerus non-primus.
 hincque adeo nulli numeri amiables oriuntur.

Exempl. 3.

§. XXXIII. Sit $m = 3$, eritque $2 = 8 \& 2 = 64$ atque
 $p = 3.8 - 1 = 23$ primus
 $q = 6.8 - 1 = 47$ primus
 $r = 18.64 - 1 = 1151$ primus.

Euleri Opuscula Tom. II.

F

Ergo

Ergo hinc numeri amiables erunt:

$$\begin{array}{l} 2^4 \cdot 23 \cdot 47 \quad \& \quad 2^4 \cdot 1151 \\ \text{sive } 17296 \quad \& \quad 18416. \end{array}$$

Exempla seqq.

§. 34. Hæc exempla cum sequentibus, in quibus exponen-
ti m majores valores tribuuntur, commodius uno conspectu ita re-
presentari poterunt.

Sit $m =$	1	2	3	4	5	6	7	8
erit $p =$	5	11	23	47	95*	191	383	767*
$q =$	11	23	47	95*	191	383	767*	1535*
$r =$	71	287*	1151	4607*	18431*	73727	294911	1179647

Ubi numeri non-primi asteriscis sunt notati: unde hinc tantum
terni numeri amiables obtinentur, nempe

$$\text{I. } \begin{cases} 2^3 \cdot 5 \cdot 7 \\ 2^3 \cdot 71 \end{cases} \quad \text{II. } \begin{cases} 2^4 \cdot 23 \cdot 47 \\ 2^4 \cdot 1151 \end{cases} \quad \text{III. } \begin{cases} 2^7 \cdot 191 \cdot 383 \\ 2^7 \cdot 73727 \end{cases}$$

Uterius autem progredi non licet, quoniam valores ipsius r
nimis sunt magni, quam ut dignosci possit, utrum sint primi nec
ne? Tabulæ namque numerorum primorum adhuc constructæ vix
ultra 100000 porriguntur.

Casus. II.

§. XXXV. Sit $k = 2$ & valores litterarum p, q, r , qui
debent esse primi, erunt:

$$\begin{array}{l} p = 5 \cdot 2^m - 1 \\ q = 20 \cdot 2^m - 1 \\ r = 100 \cdot 2^m - 1 \end{array}$$

quorum cum postremus semper sit per ternarium divisibilis, ob
 $2^{2m} = 3a + 1$ & $r = 300a + 99$, hinc nulli numeri amica-
 biles consequuntur.

Casus. III.

§. XXXVI. Ponatur $k = 3$, critque

$$p = 9 \cdot 2^m - 1$$

$$q = 72 \cdot 2^m - 1$$

$$r = 648 \cdot 2^{2m} - 1$$

quorum cum nullus necessario videatur divi-
 forem admittere, va-
 lores ipsorum p, q, r , ex valoribus simplicioribus
 ipsius m oriundos hic conjunctim repræsentabo.

$m = 1$	2	3	4	5
$p = 17$	35^*	71	143^*	287^*
$q = 143^*$	287^*	575^*	1151	2303^*
$r = 2591$	10367^*	41471^*	165887	663551

Hinc ergo, quoniam ulterius progredi non licet, nulli nu-
 meri amica- biles inveniuntur.

Casus. IV.

§. XXXVII. Ponatur $k = 4$, & sequentes tres numeri debe-
 bant esse primi.

$$p = 17 \cdot 2^m - 1$$

$$q = 272 \cdot 2^m - 1$$

$$r = 4624 \cdot 2^{2m} - 1$$

F 2

Ub



Ubi cum r semper sit multipulum ternarii, patet hinc nullos prodire numeros amicabile.

Casus. V.

§. XXXVIII. Ponatur $k=5$, & sequentes tres numeri debebunt esse primi.

$$p = 33 \cdot 2^m - 1$$

$$q = 1056 \cdot 2^{2m} - 1$$

$$r = 34848 \cdot 2^m - 1$$

Ubi statim patet casum $m=1$ esse inutilem, cum det $p=65$. Sit ergo $m=2$, fietque

$$p=131; q=4223^*, r=557567$$

ubi cum q non sit primus, & majores valores pro m ob defectum tabularum numerorum primorum examini subijci nequeant, neque hinc etiam novi numeri amicales eruantur. At vero ob eandem rationem majores valores ipsi k tribuere non licet.

Scholion.

§. XXXIX. Quoniam potestates binarii pro a positæ valorem ipsius ϵ in fractione $\frac{b}{c} = \frac{a}{2a-fa}$ unitati æqualem reddiderunt, hincque solutiones obtinere licuit, alios valores pro a , qui pariter ipsi ϵ valorem $=1$ inducant, ponam. Inter hos autem imprimis sunt notandi, qui ex hac forma $a=2(2^{n+1} + \epsilon)$ nascuntur, si quidem $2^{n+1} + \epsilon$ sit numerus primus, tum enim sit $2a =$

fa

$\sqrt{a} = e + 1$, & $\frac{b}{e} = \frac{2^{n+1} (2^{n+1} + e)}{e+1}$: si igitur $e+1$ sit divisor
 numeratoris $2^{n+1} (2^{n+1} + e)$ valor ipsius e fiet idem $= 1$.

Regula. II.

§. XL. Sit factor communis $a = 2^{n+1} (2^{n+1} + 2 - 1)$, at
 $2^{n+1} + 2 - 1$ numerus primus, erit ob $e+1 = 2$, fractio
 $\frac{e}{b} = \frac{2^{n+1} (2^{n+1} + 2 - 1)}{2^{n+1} (2^{n+1} + 2 - 1)} = 2^{n-k} (2^{n+1} + 2 - 1)$, si qui-
 dem non sit $k > n$. Hac ergo hypothesi habebimus $b = 2^{n-k}$

$(2^{n+1} + 2 - 1)$ & $e = 1$. Quadratum ergo bb in duos ejus-
 modi factores $(x-b)(y-b)$ resolvendum est, ex quibus
 non solum valores numerorum $x-1 = p$ & $y-1 = q$, sed
 etiam $xy-1 = r$ fiant numeri primi. Cujusmodi casus si ev-
 liceat, erunt numeri amabiles *apq* & *ar*. Vesum hic notandum
 est eos casus rejiciendos esse, in quibus aliquis numerorum primo-
 rum p, q, r prodit divisor ipsius a , seu æqualis $2^{n+1} + 2 - 1$, quia
 a per nullum alium numerum primum est divisibile.

Sit $n-k = m$, seu $n = m+k$, erit $a = 2^{m+k} (2^{m+k+1} + 2 - 1)$
 & $b = 2^m (2^{m+k+1} + 2 - 1)$. Jam quia $2^{m+k+1} + 2 - 1$ de-
 bet esse numerus primus, ponatur $2^{m+k+1} + 2 - 1 = f$ seu $f =$
F 3 2^m



$2^k (2^{m+1} + 1) - 1$, ut sit $a = 2^{m+k} f$ & $b = 2^m f$; erit $bb = 2^{2m} ff = (x-b)(y-b)$. Nunc ob f numerum primum, numerus $2^{2m} ff$ duplici modo in genere in duos factores resolvetur.

Priori modo fiet $(x-b)(y-b) = 2^{m-a} f \cdot 2^{m+a} f$, ideoque

$$x = 2^{m-a} f + 2^m f; \quad p = (2^{m-a} + 2^m) f - 1$$

$$y = 2^{m+a} f + 2^m f; \quad q = (2^{m+a} + 2^m) f - 1$$

$$\& r = (2^{2m+1} + 2^{2m+a} + 2^{2m-a}) ff - 1$$

qui tres numeri p, q, r debent esse primi. Posteriori modo resolutio fiet ita:

$$(x-b)(y-b) = 2^{m+a} \cdot 2^{m-a} ff, \text{ unde fit}$$

$$x = 2^{m+a} + 2^m f; \quad p = 2^{m+a} + 2^m f - 1$$

$$y = 2^{m-a} + 2^m f; \quad q = (2^{m-a} + 2^m) f - 1$$

$$\& r = (2^{2m+1} + 2^{2m+a} + 2^{2m-a}) ff - 1$$

& quoties p, q, r hoc modo prodeunt numeri primi, inde oriuntur numeri amabiles apq & ar .

Casus. I.

$$\S. \text{ XLI. Sit } k = 1, \text{ erit } a = 2^{m+1} (2^{m+2} + 1), \quad b = 2^m$$

f; erit $bb =$
 1 primum, nu-
 res resolvetur.
 + a
 f, ideoque
 m
 2) f-1
 m
 2) f-1
 2 m-a
 2) f-1
 i modo resolu-

$$2^m (2^{m+2} + 1) \text{ atque } f = 2^{m+2} + 1, \text{ qui numerus debet esse primus.}$$

Cum ergo sit $(x-b)(y-b) = 2^{2m} ff$, erit

vel

$$\begin{aligned} p &= (2^{m-a} + 2^{\frac{m}{2}}) f - 1 & p &= 2^{\frac{m+1}{2}} + 2^{\frac{m}{2}} f - 1 \\ q &= (2^{m+a} + 2^{\frac{m}{2}}) f - 1 & q &= (2^{\frac{m+1}{2}} + 2^{\frac{m}{2}}) f - 1 \\ r &= (2^{2m+1} + 2^{2m+a} + 2^{2m-a}) ff - 1 & r &= (2^{3m+1} + 2^{2m+1} + 2^{2m+a}) ff - 1 \end{aligned}$$

$m+2$

Notandum autem est, ut $2^{m+2} + 1$ sit numerus primus, exponentem $m+2$ esse oportere potestatem binarii: valores ergo ipsius m erunt: 0, 2, 6, 14, &c. At casus $m = 0$ rejici debet, ob nullum valorem ipsius a assignabilem.

Exemplum. I.

§. XLII. Sit ergo $m = 2$, ut sit $a = 8.17$ & $b = 4.17 = 68$ atque $f = 17$. Cum igitur esse debeat $(x-b)(y-b) = 4^2.17$, erit resolutione in factores instituenda:

$x - 68 =$	2	4	8	34
$y - 68 =$	8.17*	1156	578	136
$x =$	70	72	76	102
$y =$	2380	1224	646	204
$p =$	69*	71	75*	101
$q =$	2379*	1223	645*	203*
$r =$	166599*	88127*	49095*	20807

Hinc ergo nulli numeri amiables obtinentur.

Exem-

- 1
 - 1
 m+a
 ff) f-1
 inde oriuntur
 2
 +1), b=



Exemplum. 2.

§. XLIII. Sit $m = 6$, ut $a = 2^7. 257$; $b = 2^6. 257$ & $f = 257$. Cum igitur sit.

$$(x - b)(y - b) = 2^{12}. 257$$

Resolutio ita institui debet:

$x - 16448$	\equiv	$32. 257$
$y - 16448$	\equiv	$128. 257$
x	\equiv	24672
y	\equiv	49344
p	\equiv	24672
q	\equiv	49344
r	\equiv	$---$

Valores ex reliquis factoribus oriundi adhuc magis sunt magni quam ut, an primi sint nec ne, judicari possit.

Casus reliqui.

§. XLIV. Cum $f = 2^{m+k+1} + 2 - 1$ debeat esse numerus primus, quæramus primo casus simpliciores, quibus hoc evenit, cum casus nimis compositos evolvere non liceat. Sic

ergo $k = 2$, & ob $f = 2^{m+3} + 3$, valores idonei pro m erunt: 1,

3, 4: sit $k = 3$, erit $f = 2^{m+4} + 7$, & valores idonei pro m

erunt 2, 4, 6. Casu $k = 4$, est $f = 2^{m+5} + 15$, & m erit 1, vel 3 neque ulterius progredi licet.

Exempl. I.

§. XLV. Ponamus ergo $k = 2$, & $m = 1$, erit $f = 19$; & $a = 8. 19$ atque $b = 2. 19 = 38$, unde fiet

(x -

$$(x-38)(y-38) = 2^5 \cdot 19^2 = 1444$$

& resolutio dabit.

$x - 38 =$	2	4	Neuter scilicet factor assumi potest impar
$y - 38 =$	722	361	
$x =$	40		
$y =$	760	imp:	
$p =$	39^*		

Quia hic jam p non est primus, patet hinc nullos numeros amicales resultare.

Exempl. 2.

- §. XLVI. Ponamus $k=2$ & $m=3$, ut sit $f=67$ erit
 $a=32$. 67 & $b=8$. $67=536$: und sit
 $(x-536)(y-536) = 2^6 \cdot 67^2$

$x - 536 =$	268	16	reliqui valores pro p præbent numeros per 3 divisibiles, quos propterea omisi. Se- quentia exempla ad nimis magnos numeros deducunt.
$y - 536 =$	1072	17956	
$x =$	804	552	
$y =$	1608	$-$	
$p =$	803^*	1551^*	

Regula. III.

§. XLVII. Sit ut ante $a=2$ $\left(2^{\frac{n}{2} + 2 - 1} \right) \& 2^{\frac{n+1}{2} + 2 - 1}$
 $-1 = f$ numerus primus, at in fractione $\frac{b}{c} = \frac{2^{\frac{n}{2} + 2 - 1} \cdot 2^{\frac{n+1}{2} + 2 - 1}}{2^{\frac{k}{2} + 2 - 1}}$

sic $k > n$; critque $b=2^{\frac{n+1}{2} + 2 - 1} \& c=2^{\frac{k-n}{2} + 2 - 1}$. Pona-
 mus $k-n=m$, ut sit $k=m+n$, erit $a=2^{\frac{n}{2} + 2 - 1} \& 2^{\frac{n+1}{2} + 2 - 1}$;
 Euleri Opuscula Tom. II. G $k=$

($x-$



$b = 2^{n+1} + 2^{m+n} - 1 = f$ & $c = 2^m$; unde hæc habebitur
æquatio m m

$$(2^m x - b)(2^m y - b) = bb$$

Cum autem $b = f$ sit numerus primus, alia resolutio locum
non invenit præter $1, bb$: ex qua fit

$$x = \frac{1+b}{2^m} \quad \& \quad y = \frac{b(1+b)}{2^m} \quad \text{five}$$

$$x = 2^{n+1} + 2^{m+n} - 1 \quad \& \quad y = (2^{n+1} + 2^{m+n} - 1)(2^{n+1} + 2^{m+n} - 1)$$

Jam notandum est hos quatuor numeros esse oportere pri-
mos:

$f = 2^{n+1} + 2^{m+n} - 1$
 $p = x - 1$; $q = y - 1$; & $r = xy - 1$
atque necesse est ut sit $m < n+1$. Quibus conditionibus si satis-
fiat, erunt numeri amicable: apq & ar .

Casus. 1.

§. XLVIII. Sit $m = 1$, erit $f = 2^{n+2} - 1$; $x = 2^{n+1}$,
& $p = 2^{n+1} - 1$, fieri autem nequit, ut simul & f & p sit nume-
rus primus, nisi casu $n = 1$, quo vero fit $q = 27$. Ergo ex hy-
pothesi $m = 1$ nulli oriuntur numeri amicable.

Casus. 2.

§. XLIX. Sit ergo $m = 2$, ut sit $f = 3 \cdot 2^{n+1} - 1$; $x =$
 $3 \cdot 2^{n-1}$ & $y = 3 \cdot 2^{n-1} (3 \cdot 2^{n-1} - 1)$, atque $q = 2^n$.
Sequen-

Sequentes ergo quatuor numeri debent esse primi:

$$f = 3 \cdot 2^{n+1} - 1; p = 3 \cdot 2^{n-1} - 1; q = 3 \cdot 2^{n-1} \cdot 2^{n+1} - 1 \quad (3 \cdot 2^{2n-2} - 1) \\ - 1 \text{ \& } r = 9 \cdot 2^{n+1} \quad (3 \cdot 2^{2n-2} - 1) - 1, \text{ unde formantur hęc exempla;}$$

$n =$	1	2	3	4	5
$f =$	11	23	47	95*	191
$p =$	2	5	11	-	47
$q =$	32*	137	563	-	9167*
$r =$	98*	827	6767*	-	-

valet

hincque ergo ex $n = 2$, & $a = 4 \cdot 23$ nascuntur numeri amicableles:

$$\begin{cases} 4 \cdot 23 \cdot 5 \cdot 137 \\ 4 \cdot 23 \cdot 827 \end{cases}$$

Casus ceteri.

§. L. Si $m = 3$, iterum vel f vel p fit divisibile per 3, quod idem evenit si $m = 5$, vel 7, &c. Sit ergo $m = 4$; erit $f = 9 \cdot 2^{n+1} - 1$; $x = 9 \cdot 2^{n-3} - 1$; & $y = 9 \cdot 2^{n-3} \cdot 2^{n+1} - 1 \quad (9 \cdot 2^{2n-2} - 1)$ & $a = 2 \cdot f$, unde formantur hęc exempla:

$n =$	1	4	5	6
$f =$	35*	287*	575*	1151
$x =$	-	---	---	72
$y =$	-	---	---	82872
$p =$	-	---	---	71
$q =$	-	---	---	82871*
$r =$	-	---	---	-

G 2

Ne.



Neque ergo hinc neque ex majoribus valoribus ipsi m tribu-
endis numeros amiables elicere licet.

Regula. IV.

§. II. Possunt etiam alie expressiones pro factore commu-
ni a inveniri, ex quibus fractionis $\frac{b}{c}$ denominator c vel unitati,

vel potestati binarii fiat æqualis. Fingamus namque $a = 2^n (g-1)$

$(h-1)$, ut sint $g-1$ & $h-1$ numeri primi; erit $fa = 2^{n+1} (g-1)$

$gh = 2^{n+1} gh - gh$; at est $2a = 2^{n+1} gh - 2^{n+1} g - 2^{n+1} h +$

2^{n+1} unde fit

$$2a - fa = gh - 2^{n+1} g - 2^{n+1} h + 2^{n+1}$$

Ponatur $2a - fa = d$, erit $gh - 2^{n+1} (g+h) + 2^{n+1} = d$

& $(g-2^{n+1})(h-2^{n+1}) = d - 2^{n+1} + 2^{n+1}$: unde per
resolutionem in factores ejusmodi valores pro g & h elici debent,

ut $g-1$ & $h-1$ fiant numeri primi, eritque tum $a = 2^n (g-1)$
 $(h-1)$ & $\frac{b}{c} = \frac{a}{d}$.

I. Ponamus $n = 1$, erit $(g-4)(h-4) = d + 12$, ubi
ut $d + 12$ duos obtineat factores pares, sequentes prodibunt va-
lores:

Sit $d = 4$; erit $(g-4)(h-4) = 16 = 2 \cdot 8$, unde $g = 6$, $h = 12$;

$$a = 2 \cdot 5 \cdot 11 \text{ atque } \frac{b}{c} = \frac{2 \cdot 5 \cdot 11}{4} \text{ ergo } b = 5 \cdot 11 \text{ \& } c = 2.$$

Sit



Sit $d=8$; erit $(g-4)(h-4)=20=2.10$; unde $g=6, h=14$;

$$a=2.5.13, \text{ atque } \frac{b}{c} = \frac{2.5.13}{8}; \text{ ergo } b=5.13 \text{ \& } c=4.$$

Sit $d=16$; erit $(g-4)(h-4)=28=2.14$; unde $g=6, h=18$

$$a=2.5.17, \text{ atque } \frac{b}{c} = \frac{2.5.17}{16}; \text{ ergo } b=5.17 \text{ \& } c=8.$$

II. Ponamus $n=2$, erit $(g-8)(h-8)=d+56$; atque

$a=4(g-1)(h-1)$, unde sequentes casus resultant:

Sit $d=4$, erit $(g-8)(h-8)=60=6.10$, unde $g=14 \text{ \& } h=18$

$$a=4.13.17, \text{ atque } \frac{b}{c} = \frac{4.13.17}{4}; \text{ ergo } b=13.17 \text{ \& } c=1$$

Sit $d=8$, erit $(g-8)(h-8)=64=4.16$; unde $g=12, \text{ \& } h=24$

$$a=4.11.23, \text{ atque } \frac{b}{c} = \frac{4.11.23}{8}; \text{ ergo } b=11.23 \text{ \& } c=2$$

Sit $d=16$, erit $(g-8)(h-8)=72=6.12$; unde $g=14, \text{ \& } h=20$

$$a=4.13.19, \text{ atque } \frac{b}{c} = \frac{4.13.19}{16}; \text{ ergo } b=13.19 \text{ \& } c=4$$

III. Ponamus $n=3$, ut sit $a=8(g-1)(h-1)$, oportet-

bitque esse $(g-16)(h-16)=d+240$

Sit $d=4$, erit $(g-16)(h-16)=244=2.122$; unde $g=18$,

$$h=138; a=8.17.137 \text{ \& } \frac{b}{c} = \frac{8.17.137}{4}; \text{ ergo } b=17.137 \text{ \& } c=1$$

Sit $d=8$, erit $(g-16)(h-16)=248=2.124$; unde $g=18, h=140$

$$a=8.17.139 \text{ \& } \frac{b}{c} = \frac{8.17.139}{8}; \text{ ergo } b=17.139 \text{ \& } c=1$$

Sit $d=16$, erit $(g-16)(h-16)=256=4.64$; unde $g=20, h=80$

$$a=8.19.79; \frac{b}{c} = \frac{8.19.79}{16}; \text{ ergo } b=19.79 \text{ \& } c=2$$

Sit iterum $d=16$; $(g-16)(h-16)=8.3$; unde $g=24 \text{ \& } h=49$

$$a=8.23.47; \frac{b}{c} = \frac{8.23.47}{16}; \text{ ergo } b=23.47 \text{ \& } c=2.$$

G 3

Sum-



Sumtis autem hinc valoribus pro a , si numeri amica-
biles statu-
antur $a(x-1)(y-1)$ & $a(xy-1)$, ut sint $x-1$, $y-1$ & $xy-1$
numeri primi, efficiendum est ut sit $(cx-b)(cy-b) = bb$.

Exempl. 1.

§. LII. Sit $a = 2.5$. 11, erit $b = 5$. 11 = 55 & $c = 2$, un-
de fiet $(2x-55)(2y-55) = 5^2. 11^2$.

$2x-55$	1	5	25
$2y-55$	3025	605	125
x	28	30	40
y	1540	330	90
$x-1$	27*	29	39*
$y-1$	---	329*	---
$xy-1$	---	---	---

Hinc ergo nulli obti-
nentur numeri amica-
biles.

Exemplum. 2.

§. LIII. Sit $a = 2.5.13$, erit $b = 5.13 = 65$ & $c = 4$; un-
de fit $(4x-65)(4y-65) = 5^2. 13^2$.

At hic numerus $5^2. 13^2$ non resolvi potest in duos factores,
qui 65 aucti fiant per 4 divisibiles: quod idem in valore $a = 2.5.$
17 usu venit.

Exempl. 3.

LIV. Sit $a = 4.13.17$, erit $b = 13.17 = 221$ & $c = 1$ es-
seque oportet $(x-221)(y-221) = 13^2. 17^2$, unde

$x-221$	13	17	169
$y-221$	3757	---	289
$x-1$	233	237*	389
$y-1$	3977*	---	509
$xy-1$	---	---	198899

in

iles axy
& $xy - 1$
 $= bb$.

$c = 2$, ubi

in resolutione ultima sit $x - 1$ & $y - 1$ numerus primus, quæstio ergo hac redit utrum $xy - 1 = 198899$ sit numerus primus nec ne? Etiam si autem hic numerus terminum 100000 excedat, tamen demonstrare possum eum esse primum, unde numeri amicales erunt

$$\begin{cases} 4. 13. 17. 389. 509 \\ 4. 13. 17. 198899 \end{cases}$$

Scholion.

nulli obli-
meri amica-

§. LV. Numerum autem hunc 198899 esse primum inde colligo, quod observavi esse $198899 = 2 \cdot 47^2 + 441^2$, ita ut 198899 sit numerus in hac forma $2aa + bb$ contentus. Certum autem est, si quis numerus unico modo in forma $2aa + bb$ contineatur, tum eum esse primum, sin autem duplici vel pluribus modis ad formam $2a + bb$ redigi queat, tum esse compositum. Quæsi ergo utrum a numero hoc 198899 aliud quadratum duplum præter 47^2 subtrahi queat, ut residuum evadat quadratum, nullum que subducto calculo inveni: ex quo tuto conclusi hunc numerum esse primum; ideoque numeros inventos esse amicales. Ex reliquis autem valoribus ipsius a , quos exhibui, nulli reperiuntur numeri amicales.

$c = 4$; ubi

luos factores,
ore $a = 1, 5$.

Regula. V.

$1 \& c = 1 \& c$
inde

§. LVI. Possunt etiam alii numeri idonei pro a assumi, ex quibus numeros amicales eruere liceat. Cum autem pro his regula generalis tradi nequeat, aliquos tantum hic evolvam, ad quorum imitationem non erit difficile alios excogitare.

1. Sit ergo $a = 3^2$. §. 13, erit $fa = 13$. 6. 14 & ob $2a = 90$. 13 & $fa = 84$. 13, erit $2a - fa = 6$. 13 atque $\frac{b}{c} = \frac{a}{2a - fa}$

$$\frac{3^2 \cdot 5 \cdot 13}{6 \cdot 13} = \frac{15}{2} \text{ ideoque } b = 15 \& c = 2.$$

II. Sit

II. Sit $a = 3^1 \cdot 7 \cdot 13$ erit $fa = 13 \cdot 8 \cdot 14 = 16 \cdot 7 \cdot 13$ unde
ob $2a = 18 \cdot 7 \cdot 13$, erit $2a - fa = 2 \cdot 7 \cdot 13$, ideoque $\frac{b}{c} =$
 $\frac{3^1 \cdot 7 \cdot 13}{2 \cdot 7 \cdot 13} = \frac{9}{2}$; unde $b = 9$ & $c = 2$.

III. Sit $a = 3^1 \cdot 7^2 \cdot 13$, erit $fa = 13 \cdot 3 \cdot 19 \cdot 14 = 2 \cdot 3 \cdot 7 \cdot 13 \cdot$
 19 & $2a = 4 \cdot 2 \cdot 3 \cdot 7 \cdot 13$, unde $2a - fa = 4 \cdot 3 \cdot 7 \cdot 13$, ideoque $\frac{b}{c}$
 $= \frac{3^1 \cdot 7^2 \cdot 13}{4 \cdot 3 \cdot 7 \cdot 13} = \frac{21}{4}$, ergo $b = 21$ & $c = 4$.

IV. Sit $a = 3^1 \cdot 5$ erit $fa = 5 \cdot 8 \cdot 6 = 16 \cdot 3 \cdot 5$. Ergo ob
 $2a = 18 \cdot 3 \cdot 5$ erit $2a - fa = 2 \cdot 3 \cdot 5$; hincque $\frac{b}{c} = \frac{3^1 \cdot 5}{2 \cdot 3 \cdot 5} =$
 $\frac{9}{2}$ & $b = 9$ & $c = 2$.

V. Sit $a = 3^2 \cdot 5 \cdot 13 \cdot 19$, erit $fa = 13 \cdot 6 \cdot 14 \cdot 20 = 16 \cdot 3 \cdot 5 \cdot$
 $7 \cdot 13$ & ob $2a = 114 \cdot 3 \cdot 5 \cdot 13$ & $fa = 112 \cdot 3 \cdot 5 \cdot 13$ erit $\frac{b}{c} =$
 $\frac{3^1 \cdot 5 \cdot 13 \cdot 19}{2 \cdot 3 \cdot 5 \cdot 13} = \frac{3 \cdot 19}{2}$ & $b = 3 \cdot 19 = 57$ & $c = 2$.

VI. Sit $a = 3^1 \cdot 7^2 \cdot 13 \cdot 19$, erit $fa = 13 \cdot 3 \cdot 19 \cdot 14 \cdot 20 = 8 \cdot 3 \cdot$
 $5 \cdot 7 \cdot 13 \cdot 19$ & ob $2a = 42 \cdot 3 \cdot 7 \cdot 13 \cdot 19$ erit $\frac{b}{c} = \frac{3^1 \cdot 7^2 \cdot 13 \cdot 19}{2 \cdot 3 \cdot 7 \cdot 13 \cdot 19}$
 $= \frac{21}{2}$, unde fit $b = 21$ & $c = 2$.

Positis autem numeris amicabilibus $a(x-1)(y-1)$ & a
 $(xy-1)$ fieri debet $(cx-b)(cy-b) = bb$.

Exem-

Exemplum. 1.

§. LVII. Sit $b = 15$, $c = 2$; erit $a = 3^1. 5. 13$ & satisfieri oportet huic æquationi $(2x - 15)(2y - 15) = 225$:

$2x - 15$	1	5	9
$2y - 15$	25	45	25
x	8	10	12
y	120	30	20
$x - 1$	7	9*	11
$y - 1$	119*	-	19
$xy - 1$	-	-	239

Numeri ergo amicales erunt $\left\{ \begin{matrix} 3^1. 5. 13. 11. 19 \\ 3^1. 5. 13. 239 \end{matrix} \right\}$

Exempl. 2.

§. LVIII. Sit $b = 9$, $c = 2$; erit vel $a = 3^1. 7. 13$ vel $a = 3^1. 5$; & æquatio resolvenda $(2x - 9)(2y - 9) = 81$.

$2x - 9$	3	Unde cum sit $x - 1 = 5$, hic valor cum $a = 3^1. 5$ combinari nequit. Erunt ergo numeri amicales:
$2y - 9$	27	
x	6	$\left\{ \begin{matrix} 3^1. 7. 13. 5. 17 \\ 3^1. 7. 13. 107 \end{matrix} \right\}$
y	18	
$x - 1$	5	
$y - 1$	17	
$xy - 1$	107	

Exempl. 3.

§. LIX. Sit $b = 21$ & $c = 4$, erit $a = 3^1. 7^1. 13$ & æquatio resolvenda $(4x - 21)(4y - 21) = 441$.

Exem.

Euleri Opuscula Tom. II.

H

4x

$$\begin{array}{r|l}
 4x - 21 & 3 \\
 4y - 21 & 147 \\
 x & 6 \\
 y & 42 \\
 x - 1 & 5 \\
 y - 1 & 41 \\
 xy - 1 & 251
 \end{array}$$

Quia x & y debent esse numeri pares
alia resolutio locum non habet.

Ex hac ergo prodeunt numeri amica-
biles hi: $\left\{ \begin{array}{l} 3^3 \cdot 7^3 \cdot 13 \cdot 5 \cdot 41 \\ 3^3 \cdot 7^3 \cdot 13 \cdot 251 \end{array} \right\}$

Exempl. 4.

§. LX. Sit $b = 21$ & $c = 2$, erit $a = 3^3 \cdot 7^3 \cdot 13 \cdot 19$ &
æquatio resolvenda $(2x - 21)(2y - 21) = 441$.

$$\begin{array}{r|l}
 2x - 21 & 3 \\
 2y - 21 & 147 \\
 x & 12 \\
 y & 84 \\
 x - 1 & 11 \\
 y - 1 & 83 \\
 xy - 1 & 1007^*
 \end{array}$$

Quia autem valor $x - 1 = 13$
jam in valore a continetur, hinc
nulli obtineantur numeri amica-
biles.

Exemplum. 5.

§. LXI. Sit $b = 57$ & $c = 2$, erit $a = 3 \cdot 5 \cdot 13 \cdot 19$, &
æquatio resolvenda $(2x - 57)(2y - 57) = 3249$.

$$\begin{array}{r|l}
 2x - 57 & 3 \\
 2y - 57 & 1083 \\
 x & 30 \\
 y & 570 \\
 x - 1 & 29 \\
 y - 1 & 569 \\
 xy - 1 & 17099
 \end{array}$$

Hinc ergo oriuntur numeri
amicabiles hi:

$$\left\{ \begin{array}{l} 3^3 \cdot 5 \cdot 13 \cdot 19 \cdot 29 \cdot 569 \\ 3^3 \cdot 5 \cdot 13 \cdot 19 \cdot 17099 \end{array} \right\}$$

Exem.

Exemplum. 6.

§. LXII. Sit $b = 45$ & $c = 2$, erit $a = 3^4 \cdot 5 \cdot 11$, & æquatio resolvenda. $(2x - 45)(2y - 45) = 2025$

$2x - 45$	3	15
$2y - 45$	675	135
x	24	30
y	360	90
$x - 1$	23	29
$y - 1$	359	89
$xy - 1$	8639*	2699

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^4 \cdot 5 \cdot 11 \cdot 29 \cdot 89 \\ 3^4 \cdot 5 \cdot 11 \cdot 2699 \end{array} \right\}$$

Exempl. 7.

§. LXIII. Sit $b = 77$ & $c = 2$, erit $a = 3^3 \cdot 7^2 \cdot 11 \cdot 13$, & æquatio resolvenda $(2x - 77)(2y - 77) = 49 \cdot 121$.

$2x - 77$	7	11
$2y - 77$	847	539
x	42	44
y	462	308
$x - 1$	41	43
$y - 1$	461	307
$xy - 1$	19403	13551*

Hinc ergo oriuntur numeri amicabiles:

$$\left\{ \begin{array}{l} 3^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 41 \cdot 461 \\ 3^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19403 \end{array} \right\}$$

Exempl. 8.

§. LXIV. Sit $b = 105$, $c = 2$, erit $a = 3^3 \cdot 5 \cdot 7$, & æquatio resolvenda $(2x - 105)(2y - 105) = 105^2$.

$2x - 105$	3	7	15	35	Cum 102059 sit numerus primus, quia continetur in forma $8a + 3$ & unico, modo ad formam $2aa + bb$ reducitur, numeri amiables hinc orti erunt
$2y - 105$	3675	--	735	--	
x	54	56	60	70	
y	1890	--	420	--	
$x - 1$	53	55*	59	69*	
$y - 1$	1889	--	419	--	
$xy - 1$	102059	--	25199*	--	
					$\left. \begin{array}{l} 3^1 \cdot 5 \cdot 7 \cdot 53 \cdot 1889 \\ 3^1 \cdot 5 \cdot 7 \cdot 102059 \end{array} \right\}$

Scholion.

§. LXV. Numeri ergo amiables, quos hactenus ex forma *apq*; ar invenimus, sunt:

- I. $\left\{ \begin{array}{l} 2^1 \cdot 5 \cdot 11 \\ 2^1 \cdot 71 \end{array} \right\}$
- II. $\left\{ \begin{array}{l} 2^1 \cdot 23 \cdot 47 \\ 2^1 \cdot 1151 \end{array} \right\}$
- III. $\left\{ \begin{array}{l} 2^7 \cdot 191 \cdot 393 \\ 2^7 \cdot 737 \cdot 7 \end{array} \right\}$
- IV. $\left\{ \begin{array}{l} 4 \cdot 23 \cdot 5 \cdot 137 \\ 4 \cdot 23 \cdot 827 \end{array} \right\}$
- V. $\left\{ \begin{array}{l} 4 \cdot 13 \cdot 17 \cdot 389 \cdot 509 \\ 4 \cdot 13 \cdot 17 \cdot 198899 \end{array} \right\}$
- VI. $\left\{ \begin{array}{l} 3^1 \cdot 5 \cdot 13 \cdot 11 \cdot 19 \\ 3^1 \cdot 5 \cdot 13 \cdot 239 \end{array} \right\}$
- VII. $\left\{ \begin{array}{l} 3^1 \cdot 7 \cdot 13 \cdot 5 \cdot 17 \\ 3^1 \cdot 7 \cdot 13 \cdot 107 \end{array} \right\}$
- VIII. $\left\{ \begin{array}{l} 3^1 \cdot 7^1 \cdot 13 \cdot 5 \cdot 41 \\ 3^1 \cdot 7^1 \cdot 13 \cdot 251 \end{array} \right\}$

IX.

c numerus
continetur
3 & unico,
am 100+
numeri ami-
rti erunt
1589)
2059)

- IX. $\begin{cases} 3^1. 5. 13. 19. 29. 569 \\ 3^1. 5. 13. 19. 17099 \end{cases}$
 X. $\begin{cases} 3^1. 5. 11. 29. 89 \\ 3^1. 5. 11. 2699 \end{cases}$
 XI. $\begin{cases} 3^1. 7^1. 11. 13. 41. 461 \\ 3^1. 7^1. 11. 13. 19403 \end{cases}$
 XII. $\begin{cases} 3^1. 5. 7. 53. 1889 \\ 3^1. 5. 7. 102059 \end{cases}$

nus ex for-

Problema 2.

§. LXVI. *Invenire numeros amicales secundæ formæ apq
ars; positis p, q, r, s numeris primis & factore communi a dato.*

Solutio.

Cum factor communis a detur, quæatur ex eo valor fracti-
onis $\frac{b}{c} = \frac{a}{2a - fa}$ in minimis terminis, hincque erit $a : fa =$
 $b : 2b - c$. Deinde cum esse debeat $fp.fq = fr.fs$ seu $(p+1)(q+1)$
 $= (r+1)(s+1)$, ponatur uterque valor $= acxy$, & sumat-
tur:

$$p = ax - 1; q = cy - 1; r = cx - 1; s = ay - 1.$$

Ubi manifestum est hos numeros a, c, x, y, ejusmodi esse debere,
ut p, q, r, s fiant numeri primi, & numeri amicales erunt

$$a(ax - 1)(cy - 1) \text{ \& } a(cx - 1)(ay - 1)$$

Præterea vero ex natura numerorum amicabilium esse debet:

$$acxyfa = a(ax - 1)(cy - 1) + a(cx - 1)(ay - 1)$$

seu ob fa: $a = 2b - c$: b erit

IX.

H₂₃

2b



$$\left. \begin{array}{l} 2ba\epsilon xy \\ -a\epsilon xy \end{array} \right\} = \begin{array}{l} 2/a\epsilon xy - (ax - b\epsilon y + 2b) \\ -\epsilon\epsilon xy - \epsilon ay \end{array}$$

vel $\epsilon a\epsilon xy = b(a + \epsilon)(x + y) - 2b$. Unde fit

$$\epsilon a\epsilon^2 xy - b/a\epsilon^2(a + \epsilon)x - b\epsilon a\epsilon^2(a + \epsilon)y + bb(a + \epsilon)^2 = -2b\epsilon a\epsilon^2 + bb(a + \epsilon)^2$$

Quare satisfieri debet huic æquationi:

$$(\epsilon a\epsilon xy - b(a + \epsilon)(x + y) - 2b)(\epsilon a\epsilon xy - b(a + \epsilon)(x + y)) = bb(a + \epsilon)^2 - 2b\epsilon a\epsilon^2$$

Numerus ergo $bb(a + \epsilon)^2 - 2b\epsilon a\epsilon^2$ quovis casu in duo ejusmodi factores, qui sint PQ , resolvi debet, ut positus

$$x = \frac{P + b(a + \epsilon)}{\epsilon a\epsilon^2} \quad \& \quad y = \frac{Q + b(a + \epsilon)}{\epsilon a\epsilon^2}$$

hi numeri x & y non solum fiant integri, sed etiam $ax - 1$; $\epsilon y - 1$; $\epsilon x - 1$; & $ay - 1$ numeri primi. Erit igitur

$$p = \frac{P + b(a + (b - c)\epsilon)}{\epsilon\epsilon} \quad ; \quad q = \frac{Q + b\epsilon + (b - c)a}{\epsilon a}$$

$$r = \frac{P + b\epsilon + (b - c)a}{\epsilon\epsilon} \quad ; \quad s = \frac{Q + ba + (b - c)\epsilon}{\epsilon a}$$

Quovis ergo valore ipsius a proposito, unde reperitur $\frac{b}{\epsilon} =$

$\frac{a}{2a - \sqrt{a}}$, dispiendum est, utrum cum numeri a & ϵ ita assumi, tum resolutio hæc:

$$bb(a + \epsilon)^2 - 2b\epsilon a\epsilon^2 = PQ$$

ita institui queat, ut valores modo traditi pro p, q, r & s fiant numeri primi, & tales quidem, ut factor communis a nullum eorum involvat. Quoties autem his conditionibus satisfieri poterit, erunt numeri amicales: apq & ars .

Co-

Coroll.

§. LVII. Quoniam esse nequit $a = c$, pro his numeris a & c ponantur numeri simplices, hincque orientur casus sequentes:

I. Sit $a = 1$; $c = 2$; erit $PQ = 9bb - 4bc$; &

$$p = \frac{P+3b-2c}{2c}; \quad q = \frac{Q+3b-c}{c}$$

$$r = \frac{P+3b-c}{c}; \quad s = \frac{Q+3b-2c}{2c}$$

II. Sit $a = 1$; $c = 3$; erit $PQ = 16bb - 6bc$ &

$$p = \frac{P+4b-3c}{3c}; \quad q = \frac{Q+4b-c}{c}$$

$$r = \frac{P+4b-c}{c}; \quad s = \frac{Q+4b-3c}{3c}$$

III. Sit $a = 2$; $c = 3$; erit $PQ = 25bb - 12bc$

$$p = \frac{P+5b}{3c} - 1; \quad q = \frac{Q+5b}{2c} - 1$$

$$r = \frac{P+5b}{2c} - 1; \quad s = \frac{Q+5b}{3c} - 1$$

IV. Sit $a = 1$; $c = 4$; erit $PQ = 25bb - 8bc$ &

$$p = \frac{P+5b}{4c} - 1; \quad q = \frac{Q+5b}{c} - 1$$

$$r = \frac{P+5b}{c} - 1; \quad s = \frac{Q+5b}{4c} - 1$$

V. Sit

V. Sit $a = 3$; $c = 4$; erit $PQ = 49bb - 24bc$

$$p = \frac{P+7b}{4c} - 1; \quad q = \frac{Q+7b}{3c} - 1$$

$$r = \frac{P+7b}{3c} - 1; \quad r = \frac{Q+7b}{4c} - 1$$

VI. Sit $a = 1$; $c = 5$; erit $PQ = 36bb - 10bc$

$$p = \frac{P+6b}{5c} - 1; \quad q = \frac{Q+6b}{c} - 1; \quad r = \frac{P+6b}{c} - 1; \quad s = \frac{Q+6b}{5c} - 1$$

VII. Sit $a = 2$; $c = 5$; erit $PQ = 49bb - 20bc$

$$p = \frac{P+7b}{5c} - 1; \quad q = \frac{Q+7b}{2c} - 1; \quad r = \frac{P+7b}{2c} - 1; \quad s = \frac{Q+7b}{5c} - 1$$

VIII. Sit $a = 3$; $c = 5$; erit $PQ = 64bb - 30bc$

$$p = \frac{P+8b}{5c} - 1; \quad q = \frac{Q+8b}{3c} - 1; \quad r = \frac{P+8b}{3c} - 1; \quad s = \frac{Q+8b}{5c} - 1$$

IX. Sit $a = 4$; $c = 5$; erit $PQ = 81bb - 40bc$

$$p = \frac{P+9b}{5c} - 1; \quad q = \frac{Q+9b}{4c} - 1; \quad r = \frac{P+9b}{4c} - 1; \quad s = \frac{Q+9b}{5c} - 1$$

X. Sit

X. Sit $a = 1$; $c = 6$; erit $PQ = 49bb - 12bc$

$$p = \frac{P+7b}{6c} - 1; q = \frac{Q+7b}{c} - 1; r = \frac{P+7b}{c} - 1;$$

$$s = \frac{Q+7b}{6c} - 1;$$

XI. Sit $a = 5$; $c = 6$, erit $PQ = 121bb - 60bc$

$$p = \frac{P+11b}{6c} - 1; q = \frac{Q+11b}{5c} - 1; r = \frac{P+11b}{5c} - 1;$$

$$s = \frac{Q+11b}{6c} - 1;$$

Secundum hos igitur casus valores ipsius a jam ante adhibitos, quia præ ceteris ad numeros amicales inveniendos videntur apti, evolvam, ex iis autem potissimum eos eligam, qui actu ad numeros amicales deducunt.

Exemplum. I.

§. LXVIII. Sit $a = 2$; erit $b = 4$, & $c = 1$. Sumatur casus secundus quo $a = 1$, $c = 3$, ut numeri amicales sint $2'pq$ & $2'rs$, fierique debet

$$PQ = 16.16 - 6.4 = 232, \text{ atque}$$

$$p = \frac{P+16}{3} - 1; q = \frac{Q+16}{1} - 1; r = \frac{P+16}{1} - 1 \text{ \& } s =$$

$$\frac{Q+16}{3} - 1$$

factores ergo numeri 232 ita debent esse comparati, ut 16 ausi fiant per 3 divisibiles:

$$P = 2 \quad \text{Alia resolutio nulla succedit, si enim poneretur} \\ Q = 116 \quad P = 8, \text{ fieret } Q \text{ numerus impar, neque ergo } q$$

Euleri Opuscula Tom. II.

I

&

$$P + 16 = 18$$

$$Q + 16 = 132$$

$$p = 5$$

$$q = 131$$

$$r = 17$$

$$s = 43$$

& s numeri primi esse possent. Hinc ergo obtinentur hi numeri amica- biles :

$$\left\{ \begin{array}{l} 2^5 \cdot 5 \cdot 131 \\ 2^5 \cdot 17 \cdot 43 \end{array} \right\}$$

Exemplum 2.

§. LXIX. Si $a = 1$ & $c = 3$, & a potestas binarii altior: inventio numerorum amicabilium non succedit, donec perveniatur ad $a = 2^3$. Tum autem erit $b = 2^5$ & $c = 1$: atque

$$PQ = 16 \cdot 2^{16} = 6 \cdot 2^4 = 2^9 (2^{11} - 3) = 512 \cdot 2045 = 512 \cdot 5 \cdot 409;$$

$$p = \frac{P+1024}{3} - 1; q = \frac{Q+1024}{3} - 1; r = \frac{P+1024}{3} - 1; s =$$

$$\frac{Q+1024}{3} - 1$$

unde factores P & Q ita debent esse comparati, ut quaternario aucti per 3, (vel ut quoti fiant pares) per 6 sint divisibiles.

P =	2	8	20	32	80	128	320	1280
Q =	-	-	-	-	13088	8180	-	-
P + 1024 =	1026	1032	1044	1056	1104	1152	1344	2304
Q + 1024 =	-	-	-	-	14112	9204	-	-
p =	341	343*	347	-	367	383	447*	767*
q =	-	-	-	-	14111*	9203	-	-
r =	1025*	-	1043*	1055*	1103	1151	1343	2303
s =	-	-	-	-	4703	3067	-	-

Erunt ergo numeri amica- biles $\left\{ \begin{array}{l} 2^3 \cdot 383 \cdot 9203 \\ 2^3 \cdot 1151 \cdot 3067 \end{array} \right\}$

Exem-

Exempl. 3.

§. LXX. Sit $a = 2$ & $c = 3$ & fumatur $u = 3^1. 5. 13$ ut
fit $b = 15$ & $e = 2$; erit $PQ = 25. 225 = 12. 30 = 3^1. 5. 13$

$$p = \frac{P+75}{6} - 1; q = \frac{Q+75}{4} - 1; r = \frac{P+75}{4} - 1;$$

$$s = \frac{Q+75}{6} - 1.$$

unde factores P Q ejusmodi esse debent, ut ternario aucti fiant per
24 divisibiles.

P	$=$	45
Q	$=$	117
$P+75$	$=$	120
$Q+75$	$=$	192
p	$=$	19
q	$=$	47
r	$=$	29
s	$=$	31

Aliae resolutiones non inveniunt locum;
unde hinc numeri amica-
biles prode-
unt.

$$\left\{ \begin{array}{l} 3^1. 5. 13. 19. 47 \\ 3^1. 5. 13. 29. 31 \end{array} \right\}$$

Exempl. 3.

§. LXXI. Sit $a = 1$ & $c = 4$, fumatur $u = 3^1. 5$, ut fit
 $p = 9$, $e = 2$, erit $PQ = 25. 81 = 8. 18 = 9. 11. 19$ &

$$p = \frac{P+45}{8} - 1; q = \frac{Q+45}{2} - 1; r = \frac{P+45}{2} - 1;$$

$$s = \frac{Q+45}{8} - 1$$

unde P & Q ejusmodi debent esse numeri, ut quinario aucti per 8
fiant divisibiles:

1 2

$P =$

Exem-



P	=	3	19
Q	=	627	99
P + 45	=	28	64
Q + 45	=	672	144
p	=	5	7
q	=	335*	71
r	=	23	31
s	=	83	17

Hinc ergo oriuntur numeri
amicabiles:

$$\left\{ \begin{array}{l} 3^1 \cdot 5 \cdot 7 \cdot 71 \\ 3^1 \cdot 5 \cdot 31 \cdot 17 \end{array} \right\}$$

Scholion.

§. LXXII. Hæ autem operationes nimis sunt incertæ, ac plerumque plures frustra instituuntur, antequam numeri amicabiles se offerunt. Labor quoque foret vehèmenter prolixus, si singulis valoribus ipsius a , quos quidem supra exhibui, per singulos casus litterarum a & c percurrere velimus; atque nimis raro evenit, ut quatuor numeri pro p, q, r & s resultantes simul fiant primi. Tum vero etiam inventio numerorum amicabilium per determinationem rationis a & c nimis restringitur, atque dantur casus hujusmodi numerorum, in quibus ratio a & c tam est complicata, ut nulla probabili ratione eligi potuisset, cujusmodi sunt numeri amicabiles $2^4 \cdot 19 \cdot 8563$ & $2^4 \cdot 83 \cdot 2039$, ad quos hac via inveniendos ratio a : c assumi debuisset $5 : 21$ vel $1 : 102$. Hanc ob rem huic methodo nimis sterili & operose diutius non immoror, sed aliam viam aperiam, qua facilius & expeditius numeros amicabiles tam hujus secundæ formæ, quam aliarum magis compositarum investigare liceat; & quæ similis sit præcedenti, quæ tribus tantum numeris primis reperiendis absolvitur.

Problema. 2.

§. LXXIII. Invenire numeros amicabiles hujus formæ $a p q$ & a
 $f r,$

f & *c*, ubi *p*, *q*, & *r* sint numeri primi, *f* sive primus sive compositus, qui
perinde ac factor communis a sit datus.

Solutio.

Quærantur iterum ex cognito factore communi *a* valores *b*
& *c* ut sit $\frac{b}{c} = \frac{a}{2a-fa}$; & sit numeri *f* summa divisorum *ff*
 $= gh$. Cum igitur primo requiratur ut sit $fp.fq = ff.fr$, erit
 $(p+1)(q+1) = gh(r+1)$. Ponatur $r+1 = xy$, $p+1 = hx$
& $q+1 = gy$, & necesse erit, ut sint hi tres numeri primi, scilicet
 $p = hx-1$; $q = gy-1$ & $r = xy-1$. Deinde opus est, ut
sit $fpfq = gh.xyfa = a(hx-1)(gy-1) + af(xy-1) = a$
 $((gh+f)xy - hx - gy + 1 - f)$; seu $2bghxy - cghxy$
 $= b(gh+f)xy - bhx - bgy + b(1-f)$ vel
 $(bf - bgh + cgh)xy - bhx - bgy = b(f-1)$

Ponamus brevitatis gratia $bf = bgh + cgh = e$
erit $exy = ebhx = ebg = eb(f-1)$ sive:

$$(ex - bg)(xy - bh) = bbg + be(f-1)$$

Numerus ergo $bbg + be(f-1)$ in duos ejusmodi factores,
qui sint *P* & *Q* resolvi debet; ut fiant

$$x = \frac{P+bg}{e} \quad \& \quad y = \frac{Q+bh}{e} \quad \text{numeri integri, tum vero}$$

$hx-1$, $gy-1$ & $xy-1$ numeri primi. Quæ conditio, quo-
ties impleri poterit, erunt numeri amiables $a(hx-1)(gy-1)$
 $af(xy-1)$

Notandum est, neque ullum horum numerorum primorum $hx-1$,
 $gy-1$, $xy-1$, neque ullum factorem ipsius *f* divisorem esse
debere ipsius *a*: nec non f & $xy-1$ esse debere numeros primos
inter se.

Coroll. 1.

§. LXXIV. Si f sit numerus primus, uti secunda forma numerorum amicabilium postulat; erit $f+1=gh$, & propterea $f=gh-1$. Hoc ergo casu erit $e=egh-b$ & $PQ=bbgh+be$ ($gh-2$) seu $PQ=beeghh-2bgh+2bb$. Unde quaeri debent numeri x & y supra memoratis proprietatibus praediti, ut sit $x = \frac{P+bg}{e}$ & $y = \frac{Q+bh}{e}$.

Coroll. 2.

§. LXXV. His igitur formulis ita uti conveniet, ut pro a successive alii atque alii valores ex iis, quos supra exposui substituantur, atque pro singulis litteræ f varii numeri tam primi quam compositi substituantur, qui quidem ad numeros amicales inveniendos idonei videantur.

Casus. 1.

§. LXXVI. Sit $a=4$, (ex valore enim $a=2$ nullos obtineri numeros amicales observavi) eritque $b=4$ & $c=1$. Tum positis numeris amicabilibus $4pq$ & $4fr$, sit $ff=gh$, & $e=4f-3gh$. Deinde per resolutionem quaerantur factores P & Q ut sit:

$$PQ=16gh+4e(f-1)$$

Hincque eruantur numeri integri x & y , ut sit

$$x = \frac{P+4g}{e} \quad \& \quad y = \frac{Q+4h}{e}$$

& ex his deriventur valores litterarum $p=hx-1$, $q=gy-1$ & $r=xy-1$, qui si fuerint numeri primi, erunt $4pq$ & $4fr$ numeri amicales.

Exem-

Exemplum. 1.

§. LXXVII. Sit $f=3$, erit $ff=gh=4$; hincque $e=12-12=0$, unde patet ex hac hypothesi nihil obtineri.

Exempl. 2.

§. LXXVIII. Sit $f=5$, erit $ff=gh=6$; $e=20-18=2$, atque $PQ=16$. $6+8.4=128$. Jam ex $gh=6$ ponatur primo $g=2$, & $h=3$, fietque.

$$x = \frac{P+8}{2} \quad \& \quad y = \frac{Q+12}{2}$$

Quare sequentes habebuntur resolutiones:

P	$=$	2	4	8	16	32	64	
Q	$=$	64	32	16	8	4	2	
x	$=$	5	6	8	12	20	36	
y	$=$	38	22	14	10	8	7	
$p=3x-1$	$=$	19*	17	23	35*	59	107	
$q=2y-1$	$=$	--	43	27*	19	15*	13	
$r=xy-1$	$=$	--	131	111*	119*	159	251	

Hinc ergo prodeunt numeri amicabiles.
 $\{4.17.43\}$ & $\{4.5.131\}$
 $\{4.13.107\}$ & $\{4.5.251\}$

Ponatur secundo $g=1$, $h=6$, fietque:

$$x = \frac{P+4}{2} \quad \& \quad y = \frac{Q+24}{2}$$

P	$=$	2	4	8	16	32	64	
Q	$=$	64	32	16	8	4	2	
x	$=$	3	4	6	10	18	34	
y	$=$	44	28	20	16	14	13	
$p=6x-1$	$=$	17*	23	35*	59	107	203*	
$q=1y-1$	$=$	43	27*	19	15*	13	12*	
$r=xy-1$	$=$	131	111*	119*	159	251	441*	

Idem ergo prodeunt bini numeri amicabiles qui ante.

Sunt

Sunt ergo hinc numeri amica-biles:

$$\left\{ \begin{array}{l} 4. 17. 43 \\ 4. 5. 131 \end{array} \right\} \quad \& \quad \left\{ \begin{array}{l} 4. 13. 107 \\ 4. 5. 251 \end{array} \right\}$$

Exempl. 3.

§. LXXIX. Sit $f=7$, erit $ff=gh=8$; $e=28-24=4$.
& $PQ=16.8+16.6=224$.

Sit ergo primo $g=2$, $h=4$ erit

$$x = \frac{P+8}{4}; \quad y = \frac{Q+16}{4}; \quad p = 4x-1; \quad q = 2y-1; \\ r = xy-1.$$

P	4	8	28	56
Q	56	28	8	4
x	3	4	9	16
y	18	11	6	5
$4x-1$	11	15*	35*	63*
$2y-1$	35*	21*	11	9*
$xy-1$	53	42	53	79

Sit secundo $g=1$, $h=8$; erit $x = \frac{P+4}{4}$; $y = \frac{Q+32}{4}$
& $p=8x-1$; $q=y-1$; $r=xy-1$.

P	4	8	28	56
Q	56	28	8	4
x	2	3	8	15
y	22	15	10	9
$8x-1$	15*	23	63*	119*
$y-1$	21	14	9	8
$xy-1$	43	44*	79	134*

Hinc ergo nulli pro-
deunt numeri amica-
biles.

Exem-

Exemplum. 4.

§. LXXX. Sit $f = 11$, erit $gh = 12$, $e = 8$. $PQ = 16$.
 $12 + 32. 10 = 512$, vel erit $(8x - 4g)(8y - 4h) = 512$, quæ
 æquatio deprimitur ad $(2x - g)(2y - h) = 32$, qua resoluta
 erit $p = hx - 1$: $q = gy - 1$, & $r = xy - 1$. Sive autem hic
 ponatur $g = 1$, $h = 12$; sive $g = 2$, $h = 6$; sive $g = 3$, $h = 4$,
 nulli prodeunt numeri primi pro p, q & r .

Exemplum. 5.

§. LXXXI. Sit $f = 13$, erit $gh = 14$; $e = 10$; $PQ =$
 $224 + 40. 12 = 704$, & $(10x - 4g)(10y - 4h) = 704$,
 quæ deprimitur ad $(5x - 2g)(5y - 2h) = 176$. Hinc autem
 nulli alii numeri amicabile obtinentur nisi $\left\{ \begin{array}{l} 4. 5. 251 \\ 4. 13. 107 \end{array} \right\}$, qui
 jam ante (§. 78.) sunt inventi. Simul vero jam patet, etiamſi
 pro f majores numeri primi statuantur, nullos novos numeros a-
 micabiles prodire, quoniam vel p vel q fortietur valorem minorem,
 qui pro f affumi potuiffet.

Exempl. 6.

§. LXXXII. Sit $f = 5. 13$, erit $gh = 6. 14 = 84$; $e =$
 8 ; $PQ = 16. 84 + 32. 64 = 64. 53$ & $(8x - 4g)(8y - 4h)$
 $= 64. 53$ seu $(2x - g)(2y - h) = 4. 53$. Hincque invenie-
 tur in numeris primis: $p = 43$; $q = 2267$, & $r = 1187$; un-
 de erunt numeri amicabile $\left\{ \begin{array}{l} 4. 43. 2267 \\ 4. 5. 13. 1187 \end{array} \right\}$

Casus. II.

§. LXXXIII. Sit $a = 2^i = 8$, erit $b = 8$, $c = 1$, cum
 Euleri *Opuscula Tom. II.* K possi-

positis numeris amicabilibus $8pq$ & $8fr$, & $ff = gh$ erit $e = 8f - 7gh$, atque

$$(ex - 8g)(ey - 8h) = 64gb + 8e(f - 1)$$

unde casus sunt dignoscendi, quibus fiunt numeri primi

$$p = hx - 1; q = gh - 1, \text{ \& } r = xy - 1$$

Exemplum. 1.

§. LXXXIV. Sit $f = 11$ erit $gh = 12$, $e = 4$, atque

$$(4x - 8g)(4y - 8h) = 64.12 + 32.10 = 64.17 \text{ seu}$$

$$(x - 2g)(y - 2h) = 4.17 = 68.$$

Hinc autem nulli numeri amiables reperiuntur.

Exempl. 2.

§. LXXXV. Sit $f = 13$, erit $gh = 14$; $e = 6$ atque

$$(6x - 8g)(6y - 8h) = 64.14 + 48.12 = 64.23 \text{ seu}$$

$(3x - 4g)(hy - 4h) = 16.23$, verum etiam hæc hypothesis est inutilis.

Exempl. 3.

§. LXXXVI. Sit $f = 17$, erit $gh = 18$; $e = 10$, atque

$$(10x - 8g)(10y - 8h) = 64.17 + 40.16 = 64.38 \text{ seu}$$

$$(5x - 4g)(5y - 4h) = 32.19, \text{ hincque procedunt nume-}$$

ri amiables. $\left\{ \begin{array}{l} 8.23.59 \\ 8.17.79 \end{array} \right\}$

Exempl. 4.

§. LXXXVII. Magis foecunda est hypothesis $f = 11.23$

minor enim valor pro f in compositis substitui nequit, erit $gh = 12$.
24, $e = 8$ unde

$$(8x$$

gh erit = 8f

1)

imi

$(8x - 8g)(8y - 8h) = 64 \cdot 12 \cdot 24 + 64 \cdot 252$
 feu $(x - g)(y - h) = 540$. Hinc autem reperiuntur sequentes
 numeri amicabile.

$$\left\{ \begin{array}{l} 8 \cdot 383 \cdot 1907 \\ 8 \cdot 11 \cdot 23 \cdot 2543 \end{array} \right\} \left\{ \begin{array}{l} 8 \cdot 467 \cdot 1151 \\ 8 \cdot 11 \cdot 23 \cdot 1871 \end{array} \right\} \left\{ \begin{array}{l} 8 \cdot 647 \cdot 719 \\ 8 \cdot 11 \cdot 23 \cdot 1619 \end{array} \right\}$$

Hujusmodi numeris compositis proponendis multi insuper
 alii inveniuntur numeri amicabile.

e = 4, atque
 = 64. 17 seu

Scholion.

§. LXXXVIII. Ingens combinationum numerus, qui in
 hoc exemplo locum habet, ansum mihi præbuit solutionem in ali-
 am formam redigendi commodiorem. Scilicet cum sit; $e = bf -$
 $(b - c)gh$;

= 6 atque
 64. 23 seu
 a hęc hypo

$$PQ = bbgh + bc(f - 1) = (ex - bg)(ey - bh)$$

$$\text{ex formulis } x = \frac{P + bg}{e} \text{ \& } y = \frac{Q + bh}{e} \text{ eliciuntur valores}$$

$$p = \frac{hP + bgh}{e} - 1; \quad q = \frac{gQ + bgh}{e} - 1; \quad r =$$

$$\frac{PQ + b(hP + gQ) + bbgh}{ee} - 1$$

= 10, atque
 = 64. 38 seu
 leunt nume

Sit ergo ob $gh = ff$;

$$e = bf - (b - c)ff; \quad L = bbff + bc(f - 1)$$

& $MN = Lff$ erit

$$p = \frac{M + bff}{e} - 1; \quad q = \frac{N + bff}{e} - 1; \quad r = \frac{L + b(M + N) + bbff}{ee} - 1$$

f = 11. 3
 it gh = 12

& nunc quæstio eo reducitur, ut numerus Lff resolvatur in duos
 factores M & N , quorum uterque quantitate bff auctus fiat divi-
 sibilis

(12

fibilis per e , & ut quoti hinc resultantibus unitate minuti sint numeri primi. Denique oportet ut sit $r + r = \frac{(p+1)(q+1)}{ff}$ & r numerus primus. Hunc ergo calculum in nonnullis casibus illustrabo.

Casus. III.

§. LXXXIX. Sit $a = 2^4 = 16$; erit $b = 16$; $c = 1$; atque
 $e = 16f - 15ff$; $L = 256ff + 16e(f-1)$
 & $MN = Lff$
 Numeri igitur primi esse debent:
 $p = \frac{M + 16ff}{e} - 1$; $q = \frac{N + 16ff}{e} - 1$; $r = \frac{L + 256ff + 16(M+N)}{e} - 1$
 quibus inventis erunt numeri amicales;
 $16pq$ & $16fr$.

Exempl. I.

§. LXXXX. Sit $f = 17$ erit $ff = 18$; $e = 2$; $L = 1024.5$
 & $MN = 1024.5.18 = 2^9.3^5.5$
 $p = \frac{M + 288}{2} - 1$; $q = \frac{N + 288}{2} - 1$; $r = \frac{512.19 + 16(M+N)}{4} - 1$
 seu sit $M = 2m$; $N = 2n$ ut sit
 $mn = 2^8.3^5.5$ erit
 $p = m + 143$; $q = n + 143$; & $r = 8(m+n) + 243$
 qui tres numeri debent esse primi, ut numeri amicales sint
 $16pq$ & $16.17.r$.

Hoc

Hoc autem succedit duobus modis, primo si $m = 24, n = 960$
& secundò si $m = 96$ & $n = 240$; unde numeri amiables pro-
deunt:

$$\left\{ \begin{array}{l} 16. 167. 1103 \\ 16. 17. 10303 \end{array} \right\} \quad \left\{ \begin{array}{l} 16. 383. 239 \\ 16. 17. 5119 \end{array} \right\}$$

Exempl. 2.

§. LXXXI. Sit $f = 19$, erit $\frac{ff}{f} = 20, e = 4$; $L = 128.49$
& $MN = 512. 5. 49 = 2^5. 5. 7^2$. Ergo

$$p = \frac{M+320}{4} - 1; q = \frac{N+320}{4} - 1; r = \frac{128.59+16(N+N)}{16} - 1$$

feu sit $M = 4m$ & $N = 4n$ ut sit

$$mn = 32. 5. 49 = 2^5. 5. 7^2 \text{ erit}$$

$$p = m+79; q = n+79 \text{ \& } r = 4(m+n) + 711.$$

Hinc si $m = 70, n = 112$ prodeunt numeri amiables

$$\left\{ \begin{array}{l} 16. 149. 191 \\ 16. 19. 1439 \end{array} \right\}$$

Exempl. 3.

§. LXXXII. Sit $f = 23$ erit $\frac{ff}{f} = 24; e = 8, L = 256.5.7$
& $MN = 2048. 3. 5. 7 = 2^{11}. 3. 5. 7$

$$p = \frac{M+16.24}{8} - 1; q = \frac{N+16.24}{8} - 1;$$

$$r = \frac{256.59+16(M+N)}{64} - 1$$

feu sit $M = 8m; N = 8n$; & $mn = 2^5. 3. 5. 7$ erit

$$p = m+47; q = n+47; \text{ \& } r = 2(m+n) + 235$$

Hinc tres casus oriuntur; $\begin{cases} m=56; \\ n=60; \end{cases} \begin{cases} m=42; \\ n=80; \end{cases} \begin{cases} m=6 \\ n=560 \end{cases}$

K 3

&

Hoc

& numeri amicabile sunt:

$$\left\{ \begin{array}{l} 16. 103. 107 \\ 16. 23. 467 \end{array} \right\} \left\{ \begin{array}{l} 16. 89. 127 \\ 16. 23. 479 \end{array} \right\} \left\{ \begin{array}{l} 16. 53. 607 \\ 16. 23. 1367 \end{array} \right\}$$

Exemplum. 4

§. LXXXXIII. Sit $f=31$; erit $ff=32$; $L=512.31$

$$\& MN=2^4.31; p=\frac{M+16.32}{16}-1; q=\frac{N+16.32}{16}-1;$$

$$r=\frac{16(M+N)+512.47}{256}-1$$

Sit ergo $M=16m$; $N=16n$ ut sit $mn=2^4.31$ erit

$$p=m+31; q=n+31; r=m+n+93$$

Hinc autem nulli prodeunt numeri amicabile.

Exempl. 5.

§. LXXXXIV. Sit $f=47$, $ff=48$ erit $r=32$ &

$L=1024.5.7$ & $MN=2^4.3.5.7$ unde

$$p=\frac{M+16.48}{32}-1; q=\frac{N+16.48}{32}-1; \&$$

$$r=\frac{16(M+N)+1024.47}{1024}-1$$

Sit $M=32m$ & $N=32n$; ut sit $mn=2^4.3.5.7$ erit

$p=m+23$; $q=n+23$; $r=\frac{1}{2}(m+n)+46$. Ergo $m+n$ debet esse numerus impariter par, ut $\frac{1}{2}(m+n)$ fiat impar, quod evenit si vel m vel n sit impariter par. Sit

$$m=39; n=56. \text{ erunt } N. \text{ Amic. } \left\{ \begin{array}{l} 16. 53. 79 \\ 16. 47. 89 \end{array} \right\}$$

Exem-

Exempl. 6.

§. XCIV. Sit $f = 17.137$, erit $ff = 18.138 = 4.27.13$
 $= 2484$; $e = 4$; $L = 256.2484 + 64.2328 = 512.3.7.73$ &
 $MN = 1048.81.7.23.73$

$$p = \frac{M + 16.2484}{4} - 1; \quad q = \frac{N + 16.2484}{4} - 1;$$

$$r = \frac{512.2775 + 16(M + N)}{16} - 1$$

Sit $M = 4m$; $N = 4n$ erit $mn = 128.81.7.23.73$ &
 $p = m + 9935$; $q = n + 9935$ & $r = 4(m + n) + 88799$

Sed hic semper prodit valor ipsius r major quam 100000, ita
 ut difficile sit discernere, utrum sit primus nec ne.

Exempl. 7.

§. XCV. Sit $f = 17.151$ erit $ff = 18.152 = 16.9.19 =$
 2736 , $e = 32$; & $L = 1024.1967 = 1084.7.281$ atque
 $MN = 2^4.9.7.19.281$.

Sit $M = 32m$; $N = 32n$ erit $mn = 16.9.7.19.281$ &
 $p = m + 1367$; $q = n + 1367$; $r = \frac{1}{2}(m + n) + 2650$
 Sit $m = 2\mu$, $n = 8\nu$, erit $\mu\nu = 9.7.19.281$ &.
 $p = 2\mu + 1367$; $q = 8\nu + 1367$; $r = \mu + 4\nu + 2650$.

Hinc primum pater neque μ neque ν esse posse numerum for-
 mae $3a + 2$; tum μ non posse definere in 9 nec ν in 1; quibus ob-
 servatis sequentes tantum resolutiones locum habent.

$\mu \begin{cases} 3.281 \\ 21.19, 9.281 \end{cases}$	$\begin{cases} 7.19 \\ 21.281 \end{cases}$	$\begin{cases} 21.281 \\ 57 \end{cases}$	$\begin{cases} 21 \\ 57.281 \end{cases}$	$\begin{cases} 63.281 \\ 19 \end{cases}$	$\begin{cases} 3 \\ 399.281 \end{cases}$	$\begin{cases} 1 \\ 1197.281 \end{cases}$
---	--	--	--	--	--	---

quo-

Exem-



quorum ii, qui asteriscis sunt notati, excluduntur ideo, ne p, q , vel r fiat per 7 divisibile. Quarta resolutio dabit hos numeros

amicabiles $\left\{ \begin{array}{l} 16.1409.129503 \\ 16.17.151.66739 \end{array} \right\}$, si modo hic numerus 129503 est primus.

Exemplum. 8.

§. XCVI. Sit $f = 17.167$, erit $ff = 18.168 = 16.27.7 = 3024, e = 64$; $L = 2048.1797 = 2048.3.599$ &

$MN = 2^3.3^4.7.599$

fit $M = 64m$; $N = 64n$, erit $mn = 2^1.3^4.7.599$ &

$$p = m + 755; q = n + 755; r = \frac{1}{2}(m+n) + \frac{2173}{2}$$

fit $m = 2\mu$; $n = 4\nu$, erit $\mu\nu = 3^4.7.599$ &

$$p = 2\mu + 755; q = 4\nu + 755; r = \nu + \frac{\mu+1}{2} + 1086$$

Ubi patet esse oportere $\mu = 4\alpha - 1$, ne r fiat numerus par: nec $\mu = 3\alpha + 2$, nec $\nu = 3\alpha + 1$. Hinc prodeunt numeri amica-

biles $\left\{ \begin{array}{l} 16.809.51971 \\ 16.17.167.13679 \end{array} \right\}$

Casus. IV.

§. XCVII. Sit vel $a = 3^1, 5$ vel $a = 3^1.7.13$, ut fit $b = 9, e = 2$ erit $e = 9f - 7ff$; $L = 81.ff + 9e(f-1)$ & $MN = Lff$ erit

$$p = \frac{M+9ff}{e} - 1; q = \frac{N+9ff}{e} - 1;$$

$$r = \frac{9(M+N)+L+81ff}{e} - 1$$

qui

ideo, ne p,q
et hos numeros
numerus 12950;

qui numeri p,q,r si fuerint primi erunt numeri amiables.

$$\begin{Bmatrix} apq \\ afr \end{Bmatrix}$$

Exemplum.

§. XCVIII. Sit $f=7$; $ff=3$, erit $e=7$, $L=2.27.19$;

$$MN=16.27.19, \text{ erit } p = \frac{M+7^2}{7} - 1; \quad q = \frac{N+7^2}{7} - 1;$$

168=16.17.
.599 &

$$r = \frac{9(M+N)+2.27.31}{49} - 1$$

Unde posito $M=54$, $N=152$ oriuntur numeri amiables

$$\begin{array}{r} 9 \text{ \&} \\ - 2773 \\ \hline 2 \end{array}$$

$$\begin{array}{l} a. 17. 31 \\ a. 7. 71 \end{array} \quad \text{seu} \quad \left\{ \begin{array}{l} 3^1.5. 17.31 \\ 3^1.5. 7. 71 \end{array} \right\}$$

Problema. 4.

+ 1086
erius par; ne
numeri amio

§. XCIX. Invenire numeros amiables huius formæ: $egpq$, &
 ahr , ubi p,q,r sint numeri primi, at g & h five primi five compositi
dati, una cum factore communi a.

Solutio.

Ex factore communi a quærat in minimis terminis fractis
 $\frac{b}{c} = \frac{a}{2a-fa}$; deinde sit $\frac{fg}{fh} = \frac{m}{n}$; & ex prima proprietate
numerosum amabilium erit $(p+1)(q+1)fg = (r+1)fh$ seu

$$\begin{array}{l} et b=9, f= \\ LN=Lffah \end{array}$$

$$r+1 = \frac{m}{n} (p+1)(q+1). \text{ Altera vero proprietas præbet:}$$

$$(r+1)fa.fh = a(gpq+hr), \text{ vel ob } \frac{fa}{a} = \frac{2b-c}{b} \text{ erit}$$

$$(r+1)(2b-c)fh = b(gpq+hr) \text{ \& pro } r \text{ substituto valore}$$

Euleri Opuscula Tom. II.

L

m



$$m(2b-c)(p+1)(q+1)fh = b(npq + mh(p+1)(q+1) - nh^2)$$

Sit brevitatis gratia $p+1 = x$; $q+1 = y$ erit:

$$m(2b-c)xyfh = b(m^2xy + n^2xy - n^2x - ngy + ag - nh) \text{ vel}$$

$$\begin{aligned} & m^2bh \\ & nbgy - n^2gx - n^2zy = nb(h-g) \\ & - 2m^2fh \\ & + mc/fh \end{aligned}$$

Ponatur brevitatis gratia $c = b(mh + ng) - (2b - c)m/fh$
 eritque $exy - nbgy - n^2gx + nmbgg = nmbgg + nb(h-g)$ ϵ
 seu $(ex - nbgy)(y - n^2g) = nmbgg + nb(h-g)$ ϵ

Ponatur ergo $nmbgg + nb(h-g) \epsilon = MN$ fietque

$$x = \frac{M + nbgy}{\epsilon} \text{ \& } y = \frac{N + nbgy}{\epsilon} \text{ seu :}$$

$$p = \frac{M + nbgy}{\epsilon} - 1, \text{ \& } q = \frac{N + nbgy}{\epsilon} - 1; \text{ \& } r = \frac{m}{n}xy - 1$$

Qui tres numeri $p, q, \& r$ si fuerint primi, erunt numeri amiables $apq \& ahr$, dummodo utriusque factores sint primi inter se.

Coroll.

§. C. Si sint $g \& h$ numeri primi: erit $\frac{m}{n} = \frac{g+1}{h+1}$; sit

$$\begin{aligned} \text{ergo } g &= km - 1 \text{ \& } h = kn - 1; \text{ erit } fh = kn, \text{ unde fiet,} \\ r &= b(2kmn - m - n) - (2b - c)kmn = ckmn - b(m + n) \\ MN &= nb(nb(km - 1) + k(n - m) \epsilon) = (ex - bn(km - 1)) \\ (ey - bn(km - 1)) \text{ \& } p &= x - 1; q = y - 1 \text{ atque } r = \frac{m}{n} \\ xy - 1; \end{aligned}$$

Ca-

Cafus. I.

§. CI. Sit $m = 1$; $n = 3$; ergo $g = k - 1$; $h = 3k - 1$;
eritque $e = 3ck - 4b$; & $MN = 3b(3b(k-1)^2 + 2ke)$ ideoque

$$x = \frac{M + 3b(k-1)}{e}; \quad y = \frac{N + 3b(k-1)}{e}$$

ac denique $p = x - 1$; $q = y - 1$; & $r = \frac{1}{3}xy - 1$.

Exempl. I.

§. CII. Sit $a = 4$; $b = 4$; $e = 1$; erit $e = 3k - 16$; &
 $MN = 12(12(k-1)^2 + 2ke)$ & $x = \frac{M + 12(k-1)}{e}$ & $y =$

$\frac{N + 12(k-1)}{e}$. Hic poni potest

I. $k = 6$, fietque $g = 5$, $h = 17$, & $e = 2$, sed hinc nihil efficitur

II. $k = 8$, fietque $g = 7$, $h = 23$; & $e = 8$, $MN = 12(12 \cdot 49 + 128)$ seu $MN = 16 \cdot 3 \cdot 179 = (8x - 84)(8y - 84)$ ideoque $3 \cdot 179 = (2x - 21)(2y - 21)$ unde nihil pariter sequitur:

Exempl. 2.

§. CIII. Sit $a = 8$; $b = 8$, $e = 1$; erit $e = 3k - 32$; $MN =$
 $24(24(k-1)^2 + 2ke)$ seu $MN = 48(15kk - 56k + 12) = (ex - 24(k-1))(ex - 24(k-1))$

Verum ne hinc quoque quicquam concludere licet.

Cafus. II.

§. CIV. Sit $m = 3$; $n = 1$, erit $e = 3ck - 4b$; & $g = 3k - 1$;
 $h = k - 1$

$MN = b(b(3k-1)^2 - 2ke) = (ex - b(3k-1))(ey - b(3k-1))$
atque $p = x - 1$; $q = y - 1$, & $r = 3xy - 1$.



Exemplum. 1.

§. CV. Sit $a=10$, $b=5$, $c=1$, erit $e=3k--20$, &
 $5(5(3k-1)^2--2ke)=(ex-5(3k-1))(ey-5(3k-1))$
 Si hic ponatur $k=8$, fiet $5.29.89=(4x-115)(4y-115)$.
 Unde prodit $x=30$, $y=674$, $3.xy=60660$, & numeri ami-
 cabiles erunt;

$$\left\{ \begin{array}{l} 10. 23. 29. 673 \\ 10. 7. 60659 \end{array} \right\}$$

Exempl. 2.

§. CVI. Sit $a=3^1.5$, $b=9$, $c=2$, erit $e=6k--36$; &
 $(9(3k-1)^2--3ke)=(\frac{1}{3}ex-3(3k-1))(\frac{1}{3}ey-3(3k-1))$
 Jam fiat $k=8$, erit $e=12$; & $3.1523=(4x-69)(4y-69)$
 hincque oritur $x=18$, $y=398$, $3.xy=21492$ eruntque numeri
 primi; $g=23$; $h=7$; $p=17$; $q=397$; $r=21491$ & numeri ami-
 cabiles:

$$\left\{ \begin{array}{l} 3^1. 5. 23. 17. 397 \\ 3^1. 5. 7. 21491 \end{array} \right\}$$

Scholion.

§. CVII. Ex his exemplis usus hujus problematis in inve-
 niendis numeris amicabilibus satis luculenter perspicitur; sed ob
 ipsam nimiam fingendi libertatem non parum molestum est secun-
 dum præcepta hic tradita omnes casus percurrere. Cum igitur suf-
 ficiat hanc methodum tradidisse, ejusque usum monstrasse, ei pro-
 lixius non immoror, sed ad ultimam methodum, cujus ope nu-
 meros amicales eruere liceat, qua quidem sum usus, exponen-
 dam progredior. Nititur ea autem singularibus proprietatibus,
 quibus numeri ratione *summæ* divisorum gaudent, quas oblata oc-
 casione explicabo, ne plurimum lemmatum præmissio *tædium* creet.

Iis

Iis autem expositis non difficile erit plura alia problemata ad hoc genus pertinentia resolvere.

Problema 5.

§. CVIII. *Invenire numeros amicos hujus formæ: zap & zbq , ubi factores a & b sint dati, p & q numeri primi, & factor communis z investigari debeat.*

Solutio.

Sit $fa:fb = m:n$: & cum esse debeat $fa.(p+1) = fb.(q+1)$, erit $m(p+1) = n(q+1)$ Ponatur $p+1 = nx$ & $q+1 = mx$, eruntque numeri amicos $za(nx-1)$ & $zb(mx-1)$. Ubi quidem requiritur ut $mx-1$ & $nx-1$ sint numeri primi. Cum jam utriusque numeri eadem sit summa divisorum $= nxfa$, $fz = mxfb$. fz oportet ut ea sit æqualis summæ numerorum $z((na + mb)x - a - b)$. Unde obtinetur ista æquatio $\frac{z}{fz} =$

$\frac{nxfa}{(na+mb)x - a - b}$. Quo jam ex hac æquatione valor ipsius z elici

queat, fractio $\frac{nxfa}{(na+mb)x - a - b}$ ad minimos terminos reducat, quæ sit $= \frac{r}{s}$, ita ut habeatur $\frac{z}{fz} = \frac{r}{s}$: hincque se-

quentia sunt notanda. Primo esse z vel ipsi r æquale, vel ejus multiplo cuiuspiam puta kr . Priori casu si $z = r$ erit $fz = s$ ac propterea $s = fr$. Posteriori casu, si $z = kr$ erit $fz = ks = sfr$:

Verum quicquid sit k , erit $\frac{sfr}{fr} > k$, nam sfr continet omnes divisores ipsius r singulos per k multiplicatos, & insuper eos divisores ipsius r , qui non sunt per k divisibiles; eritque ergo $sfr > kfr$.

$3k-20$, &
 $5(3k-1)$
 $(4y-11)$.
2, & numeri an-

$r = 6k-36$; &
 $-3(3k-1)$
 $x = 69(4y-6)$
eruntque numeri
> 1 & numeri an-

lematis in inve-
picitur; sed al-
stum est secun-
Cum igitur sub-
strasse, ei pro-
cujus ope nu-
us, exponen-
roprietates
as oblata oc-
ædium creet.



kfr . Cum igitur sit $\frac{r}{fz} > kfr$, erit quoque $kr > kfr$ seu $r > fr$.

Quare si in fractione $\frac{r}{f}$ fuerit $r = fr$, erit $z = r$; sin autem sit $r > fr$ erit z æquale multiplo cuiusdam ipsius r . Unde patet si sit $r < fr$ æquationem $\frac{z}{fz} = \frac{r}{f}$ esse impossibilem, nequè hoc ca-

su numeros amicales inveniri posse. Deinde cum sit $\frac{fz}{z} =$

$$\frac{na+mb}{nfa} - \frac{a-b}{nxf a} = \frac{a}{fa} + \frac{b}{fb} - \frac{a-b}{nxf a}; \text{ ob } \frac{a}{fa} <$$

$1 \& \frac{b}{fb} < 1$ erit $\frac{fz}{z} < 2 - \frac{a-b}{nxf a}$, ideoque multo magis $\frac{z}{fz} > \frac{1}{2}$ ita ut z sit semper numerus deficiens. Hincque patet æquationem $\frac{z}{fz} = \frac{r}{f}$ semper ita fore comparatam, ut sit $\frac{r}{f} > \frac{1}{2}$

seu $r < 2r$. Unde si sit $fr = r$, erit $fr < 2r$, & si $r > fr$ erit multo magis $fr < 2r$. Utroque igitur casu r erit numerus deficiens. Quocirca si x tanquam numerus incognitus spectetur, propo-

sita æquatione $\frac{z}{fz} = \frac{nxf a}{(na+mb)x - a - b}$, valorem ipsius x ita determinari oportet; ut reducta fractione $\frac{nxf a}{(na+mb)x - a - b}$,

ad minimos terminos $\frac{r}{f}$, fiat r numerus deficiens, & ut sit vel $r = fr$ vel $r > fr$. Quibus conditionibus animadvertis, tam r quam s in suos factores simplices primos resolvatur, ut prodeat

huiusmodi æquatio $\frac{z}{fz} = \frac{A^a B^b C^c}{E^e F^f G^g}$, tunc autem successive
vel



vel A^{α} vel altior potestas ipsius A ponatur factor ipsius z , seu ponatur $z = P \cdot A^{\alpha + \nu}$ erit $fz = fA^{\alpha + \nu} fP$ & $\frac{z}{fz} = \frac{P A^{\alpha + \nu}}{fA^{\alpha + \nu} fP}$,

ideoque $\frac{P}{fP} = \frac{B^c C^y fA^{\alpha + \nu}}{A^c E^c F^c G^c}$. Similique modo ponatur ul-

terius $P = B^{c + \mu} Q$, & hoc pacto procedatur, donec tandem perveniatur ad æquationem hujus formæ $\frac{Z}{fZ} = \frac{u}{fu}$, ex qua habeatur $Z = u$. Sæpe quidem hæc operatio successu optato caret, sed pro quovis casu oblato facilius erit operationem hanc per exempla docere, quam per præcepta.

Exemplum. I.

§. CIX. Sit $a = 3$; $b = 1$ erit $fa = 4$; $fb = 1$; & $m = 4$; $n = 1$ ac numeri amicabiles erunt: $3(x-1)z$ & $(4x-1)z$ si sint $x-1$ & $3x-1$ numeri primi & $\frac{z}{fz} = \frac{4x}{7x-4}$. Hic autem primo patet, si 4 ex numeratore non tollatur, fore $7x-4 < \sqrt{4x}$ ob $4x = 7fx$. Ergo necesse est ut sit $7x-4$ numerus par. Ponatur $x = 4p$; erit $\frac{z}{fz} = \frac{4p}{7p-1}$. Nunc fiat $7p-1$ numerus par, ponendo $p = 2q+1$, erit $\frac{z}{fz} = \frac{2(2q+1)}{7q+3}$; & $x = 8q+4$; atque $x-1 = 8q+3$; $4x-1 = 32q+15$.

Unde



Unde q nequit esse multipulum ternarii, ne $x-1$ fiat per 3 divisibile. Erit ergo vel $q=3r+1$, vel $q=3r-1$, priori casu sit $2q+1=6r+3$, ac z deberet esse divisibile per 3, quod pariter fieri nequit, quia in altero numero quaesito $3(x-1)z$ jam inest factor 3. Sit igitur $q=3r-1$, erit $\frac{z}{fz} = \frac{2(6r-1)}{21r-4}$

atque $x=24r-4$; $x-1=24r-5$; & $4x-1=96r-17$. Cum autem z factorem 4 habere nequeat, nisi binarius ex numeratore $2(6r-1)$ tollatur, z erit divisibile per 2, & posito $z=2y$

fieri $\frac{2y}{3fy} = \frac{2(6r-1)}{21r-4}$, & $\frac{y}{fy} = \frac{3(6r-1)}{21r-4}$: ideoque

evaderet y & propterea quoque z per 3 divisibile quod fieri nequit. Hanc obrem iste binarius ex numeratore tolli debet, ponendo $r=2$; ut sit $x-1=48$; $4x-1=192$; $z=17$

eritque $\frac{z}{fz} = \frac{12s-1}{21s-2}$. Jam si s sit numerus impar ob z nu-

merum imparem, fiet quoque $fz=k(21s-2)$ numerus impar, ex quo sequitur, numerum z fore quadratum: sin autem s sit numerus par, factor communis z non erit quadratus. Evolvantur ergo ii ipsius s valores, qui efficiunt $x-1=48s-5$ & $4x-1=192s-17$ numeros primos; & dispiciatur utrum æquationi

$\frac{z}{fz} = \frac{12s-1}{21s-2}$ satisfieri queat. Sit $s=7$; erit $x-1=331$,

$4x-1=1327$ & $\frac{z}{fz} = \frac{83}{145}$. Jam cum z debeat esse qua-

dratum, ponatur $z=83^2 A$, erit $fz=367.19A$ & $\frac{A}{fA} =$

$\frac{367.19}{5.29.83}$. Nunc autem ipsius A factor statui nequit 19^2 , ob $f19^2=$

3.197

3. 127, prodiret enim 3 factor ipsius A, altioribus vero potestatibus sumendis, mox devenitur ad numeros tam grandes, ut facile pateat opus succedere non posse.

Si $s = 12$; erit $x - 1 = 571$; $4x - 1 = 2287$ & $\frac{z}{fz} = \frac{11.13}{2.125}$
 quæ neque 11^a neque 13 pro factoribus ipsius z assumendo resolvi potest.

Neque vero etiam ex majoribus valoribus pro s mihi quicquam præstare licuit.

Exempl. 2.

§. CX. Sit $a = 5$, $b = 1$; erit $fa = 6$; $fb = 1$; $m = 6$, $n = 1$ & numeri amiables erunt $5(x - 1)z$ & $(6x - 1)z$, habebiturque $\frac{z}{fz} = \frac{6x}{11x - 6}$. Quæ æquatio ut sit possibilis ex numeratore $6x$ vel binarium, vel ternarium tollere oportet, quia alioquin numerator maneret numerus redundans. Habebimus ergo duos casus evolvendos.

1. Tollatur ex numeratore ternarius ponendo $x = 3p$, erit $\frac{z}{fz} = \frac{6p}{11p - 2}$, nunc vero porro ponatur $p = 3q + 1$, eritque $\frac{z}{fz} = \frac{2(3q + 1)}{11q + 3}$ & ob $x = 9q + 3$ numeri primi esse debeat $x - 1 = 9q + 2$, & $6x - 1 = 54q + 17$, ubi patet q esse debere numerum imparem. Sit ergo $q = 2r - 1$: erit $x - 1 = 18r - 7$; $6x - 1 = 108r - 37$

$$\text{et } \frac{z}{fz} = \frac{2(6r - 2)}{22r - 8} = \frac{2(3r - 1)}{11r - 4}.$$

Evolvantur jam casus quibus $18r - 7$ & $108r - 37$ fiunt numeri primi: qui sunt:

Euleri Opera Tom. II.

M

I).^o

$$1). r=1; \text{ erit } x-1=11; 6x-1=71; \& \frac{z}{fz} = \frac{2 \cdot 2}{4} = \frac{4}{7}$$

Cum igitur hic sit $7=f_4$, erit $z=4$, & numeri amica-
 biles erunt $\left\{ \begin{array}{l} 4 \cdot 5 \cdot 11 \\ 4 \cdot 71 \end{array} \right\}$, quos quidem jam invenimus:

$$2). r=2; \text{ erit } x-1=29, 6x-1=179 \& \frac{z}{fz} = \frac{2 \cdot 5}{2 \cdot 9} = \frac{5}{9}.$$

At z factorem 5 habere nequit.

$$3). r=5; \text{ erit } x-1=83; 6x-1=503 \& \frac{z}{fz} = \frac{4 \cdot 7}{3 \cdot 17};$$

at hic $3 \cdot 17 < 4 \cdot 7$.

$$4). r=8; \text{ erit } x-1=137; 6x-1=827 \& \frac{z}{fz} = \frac{23}{2 \cdot 3 \cdot 7};$$

Ponatur $z=4 \cdot 23P$, erit $fz=24fP$ & $\frac{P}{fP} = \frac{24}{23} \cdot \frac{z}{fz} = \frac{4}{7}$;

unde $P=4$, & $z=4 \cdot 23$: quam operationem ita breviter repræ-

sento $\frac{z}{fz} = \frac{23}{2 \cdot 3 \cdot 7} \cdot \frac{4}{24} \cdot \frac{4}{7}$, unde fit $z=4 \cdot 23$ & numeri
 amica biles erunt.

$$\left\{ \begin{array}{l} 4 \cdot 23 \cdot 5 \cdot 137 \\ 4 \cdot 23 \cdot 827 \end{array} \right\}$$

Reliqui valores, quosque quidem examinavi nullos dant nume-
 ros amica biles.

II. Tollatur ex numeratore binarius, ponendo $x=2p$, erit

$$\frac{z}{fz} = \frac{6p}{11p-3}; \text{ Nunc fit } p=2q+1; \text{ erit } \frac{z}{fz} = \frac{3(2q+1)}{11q+4};$$

& numeri primi esse debebunt (ob $x=4q+2$); $x-1=4q+1$;

6x-



$$\frac{z}{\sqrt{z}} = \frac{2 \cdot 2}{4} = \frac{4}{7}$$

numeri amicali

nimus:

$$\frac{z}{\sqrt{z}} = \frac{2 \cdot 5}{2 \cdot 9} = \frac{5}{9}$$

$$\frac{z}{\sqrt{z}} = \frac{4 \cdot 7}{3 \cdot 17}$$

$$\frac{z}{\sqrt{z}} = \frac{13}{2 \cdot 37}$$

$$\frac{24}{23} \cdot \frac{z}{\sqrt{z}} = \frac{4}{7}$$

breviter repr.

4. 23 & nomen

hos dant nume-

to $x = 2p$ erit

$$= \frac{3(2q+1)}{11q+4}$$

$$-1 = 4q+11$$

$$6x$$

$6x - 1 = 24q + 11$; quare esse nequit $q = 3a - 1$. Deinde cum z non esse debeat divisibile per 5, neque $2q + 1$, neque $4q + 1$, neque $24q + 11$ per 5 debet esse divis: unde excluduntur casus $q = 5a + 2$; $q = 5a + 1$. Rejctis ergo his aliisque valoribus inutilibus ipsius q , qui pro $x - 1$ & $6x - 1$ non præbent numeros primos, calculus ita si habebit

$$q \left| \begin{array}{c} x-1 \\ 6x-1 \end{array} \right| \quad \frac{z}{\sqrt{z}} = \frac{3(2q+1)}{11q+4}$$

3	13	83	$\frac{3 \cdot 7}{37}$ nihil dat
4	17	107	$\frac{3 \cdot 9}{48} = \frac{9}{16} \left[\frac{9}{13} \frac{13}{16} \frac{13}{14} \frac{7}{8} \frac{7}{8} \right]; z = 9 \cdot 7 \cdot 13$
			vel $\frac{9}{16} \left[\frac{27}{40} \right] \frac{5}{6} \left[\frac{5}{6} \right]$ ergo $z = 27 \cdot 5$. Hic autem valor ob $a = 5$ est inutilis; erunt ergo numeri amicales
			$\left\{ \begin{array}{l} 9 \cdot 7 \cdot 13 \cdot 5 \cdot 17 \\ 9 \cdot 7 \cdot 13 \cdot 107 \end{array} \right\}$

9	37	227	$\frac{3 \cdot 19}{103}$ nihil dat
10	41	251	$\frac{3 \cdot 21}{114} = \frac{3 \cdot 7}{2 \cdot 19} \left[\frac{7}{3 \cdot 19} \right] \frac{3}{2 \cdot 7} \left[\frac{3}{13} \right] \frac{13}{14} \left[\frac{13}{14} \right]$
			Ergo $z = 3 \cdot 7 \cdot 13 \cdot 5 \cdot 41$ & numeri amicales erunt
			$\left\{ \begin{array}{l} 3 \cdot 7 \cdot 13 \cdot 5 \cdot 41 \\ 3 \cdot 7 \cdot 13 \cdot 251 \end{array} \right\}$

M 2

18

$$1). r=1; \text{erit } x-1=11; 6x-1=71; \& \frac{z}{fz} = \frac{2 \cdot 2}{4} = \frac{4}{7}$$

Cum igitur hic fit $7=f_4$, erit $z=4$, & numeri amicabiles erunt $\left\{ \begin{array}{l} 4 \cdot 5 \cdot 11 \\ 4 \cdot 71 \end{array} \right\}$, quos quidem jam invenimus:

$$2). r=2; \text{erit } x-1=29, 6x-1=179 \& \frac{z}{fz} = \frac{2 \cdot 5}{2 \cdot 9} = \frac{5}{9}.$$

At z factorem 5 habere nequit.

$$3). r=5; \text{erit } x-1=83, 6x-1=503 \& \frac{z}{fz} = \frac{4 \cdot 7}{3 \cdot 17};$$

at hic $3 \cdot 17 < 4 \cdot 7$.

$$4). r=8; \text{erit } x-1=137; 6x-1=827 \& \frac{z}{fz} = \frac{2 \cdot 3}{2 \cdot 3 \cdot 7};$$

Ponatur $z=4 \cdot 23P$, erit $fz=24fP$ & $\frac{P}{fP} = \frac{24}{23} \cdot \frac{z}{fz} = \frac{4}{7}$, unde $P=4$, & $z=4 \cdot 23$: quam operationem ita breviter repræ-

sento $\frac{z}{fz} = \frac{23}{2 \cdot 3 \cdot 7} \cdot \frac{23}{24} \cdot \frac{4}{7} \cdot \frac{4}{7}$, unde fit $z=4 \cdot 23$ & numeri amicabiles erunt.

$$\left\{ \begin{array}{l} 4 \cdot 23 \cdot 5 \cdot 137 \\ 4 \cdot 23 \cdot 827 \end{array} \right\}$$

Reliqui valores, quosque quidem examinavi nullos dant numeros amicabiles.

II. Tollatur ex numeratore binarius, ponendo $x=2p$ erit

$$\frac{z}{fz} = \frac{6p}{11p-3}; \text{ Nunc fit } p=2q+1; \text{ erit } \frac{z}{fz} = \frac{3(2q+1)}{11q+4} \& \text{ numeri primi esse debebunt (ob } x=4q+2); x-1=4q+1;$$

$6x$

$6x-1=24q+11$: quare esse nequit $q=3\alpha-1$. Deinde cum z non esse debeat divisibile per 5, neque $2q+1$, neque $4q+1$, neque $24q+11$ per 5 debet esse divis: unde excluduntur casus $q=5\alpha+2$; $q=5\alpha+1$. Rejctis ergo his aliisque valoribus inutilibus ipsius q , qui pro $x=1$ & $6x-1$ non præbent numeros primos, calculus ita se habebit

$$q \left| \begin{array}{c} x-1 \\ 6x-1 \end{array} \right| \quad \frac{z}{fz} = \frac{3(2q+1)}{11q+4}$$

3	13	83	$\frac{3 \cdot 7}{37}$ nihil dat
4	17	107	$\frac{3 \cdot 9}{48} = \frac{9}{16} \left[\frac{9}{13} \frac{13}{16} \frac{13}{14} \frac{7}{8} \frac{7}{8} \right]; z = 9 \cdot 7 \cdot 13$ vel $\frac{9}{16} \left[\frac{27}{40} \right] \frac{5}{6} \left[\frac{5}{6} \right]$ ergo $z = 27 \cdot 5$. Hic autem valor ob $\alpha = 5$ est inutilis; erunt ergo numeri amiables $\left\{ \begin{array}{l} 9 \cdot 7 \cdot 13 \cdot 5 \cdot 17 \\ 9 \cdot 7 \cdot 13 \cdot 107 \end{array} \right\}$

9	37	227	$\frac{3 \cdot 19}{103}$ nihil dat
10	41	251	$\frac{3 \cdot 21}{114} = \frac{3 \cdot 7}{2 \cdot 19} \left[\frac{7^3}{3 \cdot 19} \right] \frac{3^3}{2 \cdot 7} \left[\frac{3^3}{13} \frac{13}{14} \frac{13}{14} \right]$ Ergo $z = 3^3 \cdot 7^3 \cdot 13$ & numeri amiables erunt $\left\{ \begin{array}{l} 3^3 \cdot 7^3 \cdot 13 \cdot 5 \cdot 41 \\ 3^3 \cdot 7^3 \cdot 13 \cdot 251 \end{array} \right\}$

18	73	443	$\frac{3 \cdot 37}{262} = \frac{3 \cdot 37}{2 \cdot 101}$	nihil dat.
24	97	587	$\frac{3 \cdot 49}{268} = \frac{3 \cdot 49}{4 \cdot 67}$	nihil dat.
28	113	683	$\frac{3 \cdot 57}{312} = \frac{9 \cdot 19}{8 \cdot 39} = \frac{3 \cdot 19}{8 \cdot 13}$	nihil dat.
34	137	827	$\frac{3 \cdot 69}{378} = \frac{23}{2 \cdot 21} = \frac{23}{2 \cdot 3 \cdot 7} \cdot \frac{23}{24} \cdot 4 \cdot \frac{4}{7}$	$z =$
			4. 23; ut ante	
39	157	947	$\frac{3 \cdot 79}{433}$	nihil dat.
45	181	1091	$\frac{3 \cdot 91}{499} = \frac{3 \cdot 7 \cdot 13}{499}$	
48	193	1163	$\frac{3 \cdot 97}{532} = \frac{3 \cdot 97}{4 \cdot 7 \cdot 19} = \frac{3 \cdot 97}{4 \cdot 133} \cdot \frac{97}{2 \cdot 7} \cdot \frac{7}{2 \cdot 19} \cdot \frac{7}{3 \cdot 19} \cdot 2 \cdot 7$	3^2
			$\frac{3^2}{13} \cdot \frac{13}{14}$	

Ergo $z = 3^2 \cdot 7^2 \cdot 13 \cdot 97$ & numeri amicabiles sunt

$\left\{ \begin{array}{l} 3^2 \cdot 7^2 \cdot 13 \cdot 97 \cdot 5 \cdot 193 \\ 3^2 \cdot 7^2 \cdot 13 \cdot 97 \cdot 1163 \end{array} \right\}$

49	197	1187	$\frac{3 \cdot 99}{543} = \frac{9 \cdot 11}{181}$
60	241	1451	$\frac{3 \cdot 121}{664} = \frac{3 \cdot 11^2}{8 \cdot 83}$

Ergo $z = 3^3 \cdot 7 \cdot 13 \cdot 41 \cdot 163$ & numeri am-
cibiles erunt

Exempl. 3.

§. CXI. Sit $a = 7$; $b = 1$; erit $fa = 8$, $fb = 1$; $m = 8$,
 M_3 $n = 1$

$n = 1$ & numeri amicales: $7(x-1)z$ & $(8x-1)z$, ex-
 stente $\frac{z}{fz} = \frac{8x}{15x-8}$. Ac primo quidem x debet esse nume-
 us par: ponatur ergo $x = 2p$; erit $x-1 = 2p-1$; $8x-1$
 $= 16p-1$ & $\frac{z}{fz} = \frac{8p}{15p-4}$: quæ æquatio est impossibilis,
 nisi potestas binarii in numeratore deprimatur, quia $15p-4 <$
 $8p$. Ergo fiat $p = 4q$, ut sit $x = 8q$; $x-1 = 8q-1$; $8x-$
 $1 = 64q-1$ & $\frac{z}{fz} = \frac{8q}{15q-1}$. Nunc sit $q = 2r+1$; erit
 $\frac{z}{fz} = \frac{4(2r+1)}{15r+7}$ & $x-1 = 16r+7$; $8x-1 = 128r$
 $+ 63$, quorum numerorum ut neuter sit per 3 divisibilis, ne-
 que erit $r = 3a-1$, neque $r = 3a$. Sit ergo $r = 3s+1$; erit
 $\frac{z}{fz} = \frac{4(6s+3)}{45s+22}$ seu $\frac{z}{fz} = \frac{4 \cdot 3(2s+1)}{45s+22}$ & $x-1 = 48s+$
 23 ; $8x-1 = 384s+191$.

Nunc vel ternarius vel quaternarius ex numeratore tolli debet.
 At ternarius tolli nequit, quia denominator nunquam per 3 est
 divisibilis; tollatur ergo quaternarius, ad quod pono $s = 2t$ erit,
 que $\frac{z}{fz} = \frac{2 \cdot 3(4t+1)}{45t+11}$; nunc sit $t = 2u-1$; erit
 $\frac{z}{fz} = \frac{3(8u-3)}{45u-17}$: at est $s = 4u-2$, ideoque numeri pri-
 mi esse debent $x-1 = 192u-73$; $8x-1 = 1536u-$
 577

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$x-1$	$8x-1$	$\frac{z}{fz}$
$1)u=5$	887 7103	$\frac{3 \cdot 37}{208} = \frac{3 \cdot 37}{16 \cdot 13}$
		$\frac{37}{2 \cdot 19} \frac{3 \cdot 19}{8 \cdot 13} \frac{19}{4 \cdot 5}$
		$\frac{3 \cdot 5}{2 \cdot 13} \frac{5}{2 \cdot 3} \frac{3^3}{13}$

Ergo $z=3^3 \cdot 5 \cdot 19 \cdot 37$ & numeri amicabiles erunt:

$$\left\{ \begin{array}{l} 3^3 \cdot 5 \cdot 19 \cdot 37 \cdot 7 \cdot 887 \\ 3^3 \cdot 5 \cdot 19 \cdot 37 \cdot 7103 \end{array} \right\}$$

$u=11$	2039 16319	$\frac{3 \cdot 5 \cdot 17}{4 \cdot 107}$
$u=13$	2423 19391	$\frac{3 \cdot 101}{8 \cdot 71}$
$u=26$	4919 39359	$\frac{3 \cdot 205}{1153}$

Exemplum. 4.

§. CXH. Sit $a=11$, $b=1$, erit $fz=m=11$; $fz=n=1$; numeri quæfiti $11(x-1)z$ & $(12x-1)z$; atque

$$\frac{z}{fz} = \frac{12x}{23x-12}. \quad \text{Hic ex numeratore vel 3 vel 4 tolli debet:}$$

I. Tollatur 3; ponatur $x=3p$; erit $\frac{z}{fz} = \frac{12p}{23p-4}$; &

$$p=$$

$p = 3q - 1$, erit $\frac{z}{fz} = \frac{4(3q-1)}{23q-9}$; & ob $x = 9q - 3$, q debet esse impar. Sit $q = 2r + 1$, ut sit $x = 18r + 6$, erit $\frac{z}{fz} = \frac{4(6r+2)}{46r+14} = \frac{4(3r+1)}{23r+7}$, & $x - 1 = 18r + 5$; $12x - 1 = 216r + 71$.

r	$x - 1$	$12x - 1$	$\frac{z}{fz}$
0	5	71	$\frac{4}{7}$; $z = 4$; num. amic. $\left\{ \begin{array}{l} 4, 11, 5 \\ 4, 71 \end{array} \right.$
2	41	503	$\frac{4 \cdot 7}{53}$
3	59	719	$\frac{4 \cdot 10}{76} = \frac{2 \cdot 5}{19}$ imp.
6	113	1367	$\frac{4 \cdot 19}{145} = \frac{4 \cdot 10}{3 \cdot 29}$ imp.
7	131	1583	$\frac{4 \cdot 22}{168} = \frac{11}{21} = \frac{11}{3 \cdot 7} \left[\begin{array}{c} 11 \\ 12 \end{array} \right] \frac{4}{7}$ sed

ob factorem 11 hic valor z non valet.

II. Tollatur factor 4, ac ponatur $x = 4p$; ut fiat $\frac{z}{fz} = \frac{12p}{23p-3}$. Jam sit $p = 4q + 1$, erit $\frac{z}{fz} = \frac{3(4q+1)}{23q+5}$, & ob

ob $x = 16q + 4$ numeri primi esse debent $x - 1 = 16q + 3$ &
 $12x - 1 = 192q + 47$, hinc excluduntur valores $q = 3a$.

q	$x - 1$	$12x - 1$	$\frac{z}{fz}$
0	3	47	$\frac{3}{5}$ imposs.
1	19	239	$\frac{3 \cdot 5}{4 \cdot 7} \left[\frac{5}{2 \cdot 3} \right] \frac{3^1}{14} \left[\frac{3^1}{13} \right] \frac{13}{14}; z = 3^1 \cdot 5.$ 13 & numeri amicales erunt $\left\{ \begin{array}{l} 3^1 \cdot 5 \cdot 13 \cdot 11 \cdot 19 \\ 3^1 \cdot 5 \cdot 13 \cdot 239 \end{array} \right\}$

13	211	2543	$\frac{3 \cdot 53}{16 \cdot 19} \left[\frac{53}{2 \cdot 27} \right] \frac{81}{8 \cdot 19} \left[\frac{243}{4 \cdot 7 \cdot 13} \right] \frac{7 \cdot 13}{2 \cdot 3 \cdot 19} \left[\frac{13}{2 \cdot 7} \right]$ $\frac{7^1}{3 \cdot 19} \left[\frac{7^1}{3 \cdot 19} \right]$ Ergo $z = 3^1 \cdot 7^1 \cdot 13 \cdot 53$ & numeri amicales erunt $\left\{ \begin{array}{l} 3^1 \cdot 7^1 \cdot 13 \cdot 53 \cdot 11 \cdot 211 \\ 3^1 \cdot 7^1 \cdot 13 \cdot 53 \cdot 2543 \end{array} \right\}$
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Exemplum. 5.

§. CXIII. Sit $a = 5$; $b = 17$; & numeri amicales

$$5(3x - 1)z \text{ \& } 17(x - 1)z; \text{ erit } \frac{z}{fz} = \frac{18x}{32x - 22} = \frac{9x}{16x - 11};$$

Euleri Opuscula Tom. II. N Cum

Cum x debeat esse numerus par, ponatur $x = 2p$, erit $\frac{z}{fz} =$

$\frac{18p}{32p - 11}$, & ex numeratore $18p$ vel factor 2 vel 3, tolli debet, ne sit numerus redundans. At factor 2 tolli nequit; tollatur ergo factor 9. Ad hoc ponatur $p = 9q + 4$, ut sit $x = 18q + 8$ & $x - 1 = 18q + 7$ & $3x - 1 = 54q + 23$, erit $\frac{z}{fz} = \frac{2(9q + 4)}{32q + 13}$.

q	$x - 1$	$3x - 1$	$\frac{z}{fz}$
0	7	23	$\frac{8}{13}$ imposs.
2	43	131	$\frac{4 \cdot 11}{7 \cdot 11} = \frac{4}{7}$; $z = 4$ & N. A. $\{4 \cdot 5 \cdot 131\}$ $\{4 \cdot 17 \cdot 43\}$
4	79	239	$\frac{16 \cdot 5}{3 \cdot 47}$
5	97	293	$\frac{2 \cdot 49}{173}$
17	313	941	$\frac{2 \cdot 157}{557}$
19	349	1049	$\frac{2 \cdot 5 \cdot 7}{27 \cdot 23}$
20	367	1103	$\frac{16 \cdot 23}{653}$
24	439	1319	$\frac{8 \cdot 5 \cdot 11}{781}$ inut. $= \frac{8 \cdot 5}{71}$

Exem-

Exempl. 6.

erit $\frac{z}{fz} =$
 tolli debet,
 tollatur er-
 $13q+8 \&$
 $2(9q+4)$
 $32q+13$

§. CXIV. Sit $a = 37$ & $b = 227$, erit $fa = 38$, $fb = 228$,
 & $\frac{m}{n} = \frac{1}{6}$; unde si numeri amicales sint $37(6x-1)z$ &
 $227(x-1)z$, fiet $\frac{z}{fz} = \frac{6 \cdot 38x}{449x-254} = \frac{4 \cdot 3 \cdot 19x}{449x-254}$: Ubi
 cum x debeat esse numerus par, ponatur $x = 2p$, ut numeri primi
 esse debeant $x-1 = 2p-1$ & $6x-1 = 12p-1$, eritque
 $\frac{z}{fz} = \frac{4 \cdot 3 \cdot 19p}{449p-132}$. Nunc ex numeratore vel factor 4 vel factor
 3 tolli debet.

I. Tollatur factor 3, ad hoc ponatur $p = 3q$, ut sit

$\frac{z}{fz} = \frac{4 \cdot 3 \cdot 19q}{449q-44}$; nunc fiat $q = 3r+1$, eritque;
 $\frac{z}{fz} = \frac{4 \cdot 19(3r+1)}{449r+135}$. & $p = 9r+3$; $x-1 = 18r+5$
 $6-1 = 108r+35$

r	$x-1$	$6x-1$	$\frac{z}{fz}$
2	41	251	$\frac{4 \cdot 19 \cdot 7}{1033}$
3	59	359	$\frac{4 \cdot 19 \cdot 10}{1481} = \frac{4 \cdot 5}{3 \cdot 13}$
6	113	683	$\frac{4 \cdot 19 \cdot 19}{3 \cdot 23 \cdot 41}$
13	239	1439	$\frac{4 \cdot 19 \cdot 49}{4 \cdot 1493}$

17	311	1871	$\frac{16.13.19}{8.971}$
22	401	2411	$\frac{4.19.67}{10013} = \frac{4.67}{17.31} \left[\frac{67}{4.17} \right] \frac{16}{31} \left[\frac{16}{31.} \right]$
			$z = 16.67$
			Num. Amicab. $\left\{ \begin{array}{l} 16.67.37.2411 \\ 16.67.227.401 \end{array} \right\}$
117	2111	12671	$\frac{4.19.352}{52668} = \frac{128.11.19}{4.7.9.11.19} = \frac{32}{63}; z = 32$
			& num. amic. $\left\{ \begin{array}{l} 32.37.12671 \\ 32.227.2111 \end{array} \right\}$

II. Tollatur factor 4; ponatur $p = 47$; erit $\frac{z}{fz} = \frac{4.3.19q}{449q-33}$
 nunc sit $q = 4r+1$, erit $p = 16r+4$; $x-1 = 32r+7$;
 $6x-1 = 192r+47$ atque $\frac{z}{fz} = \frac{3.19(4r+1)}{449r+104}$.

r	$x-1$	$6x-1$	$\frac{z}{fz}$
0	7	47	$\frac{3.19}{8.13} \left[\frac{19}{4.5} \right] \frac{3.5}{2.13} \left[\frac{5}{2.3} \right] \frac{3}{13}; z = 3^3.5.19$
			& num. Am. $\left\{ \begin{array}{l} 3^3.5.19.37.47 \\ 3^3.5.19.227.7 \end{array} \right\}$

2	71	431	$\frac{9.19}{2.167}$
8	263	1583	$\frac{3.19.53}{16.3.7.11} = \frac{3.19}{16.7} \left \frac{19}{4.5} \right \left \frac{3.5}{4.7} \right \left \frac{5}{2.3} \right $
			$\frac{3^3}{2.7} \left \frac{3^3}{13} \right \frac{13}{14}$
			$z = 3^3.5.13.19 \&$
			Num. Amic. $\left\{ \begin{array}{l} 3^3.5.13.19.37.1583 \\ 3^3.5.13.19.227.263 \end{array} \right\}$

15	487	2927	$\frac{3.19.61}{7.977}$
23	743	4463	$\frac{9.19.31}{9.19.61} = \frac{31}{61}$
26	839	5039	$\frac{3.19.105}{2.3.13.151} = \frac{3.5.7.19}{2.13.151}$
30	967	5807	$\frac{3.19.11}{2.617}$
41	1319	7919	$\frac{3.19.165}{9.121.17} = \frac{5.19}{11.17}$

Exempl. 7.

§. CXV. Sit $a = 79$; $b = 11.19 = 209$; $fa = 80$; $fb = 240$ erit $m = 1, n = 3$, & numeri amicales sint $79(3x-1)z$ & $11.19(x-1)z$, erit $\frac{z}{fz} = \frac{240x}{416x-288} = \frac{120x}{223x-144}$,
 $N \quad 3 \quad \text{Sit}$

Sit $x = 2p$ erit $\frac{z}{fz} = \frac{120p}{223p-72}$, & numeri primi esse debent
 $2p-1$ & $6p-1$

Nunc autem ex numeratore 120p vel factor 8 vel 3 tolli debet.

I. Tollatur factor 3; sit $p = 9q$ erit $\frac{z}{fz} = \frac{120q}{223q-8}$

& fiat $q = 3r-1$, ut sit $\frac{z}{fz} = \frac{40(3r-1)}{223r-77}$; $p = 27r-9$ & $x-1 = 54r-19$; ac $3x-1 = 162r-55$.

Nunc autem ob 40 numerum redundantem vel 5 vel 4 tolli debet.

a) Tollatur 5, sitque $r = 5s-1$; erit $\frac{z}{fz} = \frac{8(15s-4)}{223s-60}$

& numeros primos esse oportet $x-1 = 470s-73$; $3x-1 = 810s-217$. Ac ne ternarius denuo in numeratorem intret, excludendi sunt casus $s = 3a-1$. Hinc autem nihil invenitur.

c) Cum sit $\frac{z}{fz} = \frac{40(3r-1)}{223r-77}$ tollatur 4: sitque $r =$

$4s-1$; erit $\frac{z}{fz} = \frac{10(12s-1)}{223s-75} = \frac{40(3s-1)}{223s-75}$; sit

porro $s = 4t+1$ erit $\frac{z}{fz} = \frac{10(12t+2)}{223t+37} = \frac{20(6t+1)}{223t+37}$. Sit

porro $t = 2u-1$; erit $\frac{z}{fz} = \frac{10(12u-5)}{223u-93}$; & ob $r = 16s-$

$+3 = 32u-13$; erit $x-1 = 1728u-721$
 $3x-1 = 5184u-2161$

At hos numeros non reddit primos minor valor ipsius u quam 16,

unde sit $\frac{z}{fz} = \frac{2 \cdot 11 \cdot 17}{5 \cdot 139}$, qui ob factorem 11 est inutilis.

II. Er-

debet.

debet.

120 q

34 r - 8

27 r -

li debet.

r - 4

r - 60

3 x - 1

ret, ex-

itur.

ue r =

-1) Sit

37

= 161

m 16,

I. Er-

II. Ergo ex æq. $\frac{z}{fz} = \frac{120q}{223p-72}$ tollatur factor 8. Ponatur

$p = 8q$, erit $\frac{z}{fz} = \frac{120q}{223q-9}$ & nunc sit $q = 8r - 1$ erit

$\frac{z}{fz} = \frac{3 \cdot 5(8r-1)}{223r-29}$; at ob $p = 64r - 8$, erit

$x - 1 = 128r - 17$; $3x - 1 = 384r - 49$

Unde valores excluduntur $r = 3a + 1$; & $r = 5a + 1$.

r	x-1	3x-1	$\frac{z}{fz}$
2	239	719	$\frac{3 \cdot 5^3}{139}$
3	367	1103	$\frac{3 \cdot 23}{128} \left[\frac{23}{8 \cdot 3} \right] \frac{3^3}{16} \left[\frac{3^3}{13} \right] \frac{13}{16} \left[\frac{13}{14} \right] \frac{7}{8}$
			Ergo $z = 3^3 \cdot 7 \cdot 13 \cdot 23$ vel
			$\frac{3 \cdot 23}{128} \left[\frac{23}{8 \cdot 3} \right] \frac{3^3}{16} \left[\frac{3^3}{8 \cdot 5} \right] \frac{5}{6}$; ergo $z =$

$3^3 \cdot 5 \cdot 23$

& numeri amicales erunt.

$\left\{ \begin{matrix} 3^3 \cdot 7 \cdot 13 \cdot 23 \cdot 79 \cdot 1103 \\ 3^3 \cdot 7 \cdot 13 \cdot 23 \cdot 11 \cdot 19 \cdot 367 \end{matrix} \right\}$ vel $\left\{ \begin{matrix} 3^3 \cdot 5 \cdot 23 \cdot 79 \cdot 1103 \\ 3^3 \cdot 5 \cdot 23 \cdot 11 \cdot 19 \cdot 367 \end{matrix} \right\}$

Exempl. 8.

§. CXVI. Sit $a = 17 \cdot 19$; $b = 11 \cdot 59$; erit $fa = 18 \cdot 20$,
 $fb =$



$fb = 12.60$, & $m = 1$, $n = 2$. Si ergo num. am. ponantur.

$$17.19(2x-1)z \text{ erit } \frac{z}{fz} = \frac{720x}{1295x-972}; \text{ Sit } x = 2p$$

$$11.59(x-1)z \text{ erit } \frac{z}{fz} = \frac{720p}{1295p-486}; \text{ atque } \frac{x-1=2p-1}{2x-1=4p-1}; \text{ quorum}$$

ut neuter sit divisibilis per 3, debet esse $p = 3q$, ut sit

$$\frac{z}{fz} = \frac{720q}{1295q-162}; \text{ \& } \frac{x-1=6q-1}{2x-1=12q-1}$$

Tollatur ex num. factor 16, sitque $q = 2r$ erit

$$\frac{z}{fz} = \frac{720r}{1295r-81}; \text{ nunc sit } r = 16s-1 \text{ erit}$$

$$\frac{z}{fz} = \frac{45(16s-1)}{1295s-86} \text{ \& } \frac{x-1=192s-13}{2x-1=384s-25}$$

Sit $s = 1$; erit $x-1 = 179$; $2x-1 = 359$ &

$$\frac{z}{fz} = \frac{45.15}{1209} = \frac{225}{403} = \frac{3^2.5^2}{13.31} \left[\frac{3^2}{13} \right] \frac{5^2}{31} \left[\frac{5^2}{31} \right]$$

Ergo $z = 3^2.5^2$ & numeri amiables erunt

$$\left\{ \begin{array}{l} 3^2.5^2.17.19.359 \\ 3^2.5^2.11.59.179 \end{array} \right\}$$

Scholion.

§. CXVII. Hæc ultima methodus in problemate 5. expofita prorfus diverfa eft a methodo præcedente, quam problemata 4 priora complectuntur: dum in hac factor communis quæritur, in illa autem datur. Utraque tamen fingulari præftantiæ genere eft prædita, ut altera fine subsidio alterius non fatis apta fit ad multitudinem

porantur.

$x=2p$

quorum

lit

tudinem numerorum amicabilium augendam. Posterior enim methodus suppeditat ejusmodi factores communes, quos ad usum prioris vix suspicari licuisset: prior vero suggerit reliquos factores huic instituto idoneos. Ceterum cuncta, quæ hic tradidi, specimen continent methodi summæ incertæ, quam, quantum licuit ad regulas algebraicas reduxi, ut vaga tentandi incertitudo restringeretur. Coronidis ergo loco ultra sexaginta numerorum amicabilium paria subjungam, quos his methodis elicui.

Catalogus numerorum amicabilium.

I { $\begin{matrix} 2^1.5.11 \\ 2^1.71 \end{matrix}$ }	II { $\begin{matrix} 2^1.23.47 \\ 2^1.1151 \end{matrix}$ }	III { $\begin{matrix} 2^1.191.383 \\ 2^1.73727 \end{matrix}$ }
IV { $\begin{matrix} 2^1.23.5.137 \\ 2^1.23.827 \end{matrix}$ }	V { $\begin{matrix} 3^1.7.13.5.17 \\ 3^1.7.13.107 \end{matrix}$ }	
VI { $\begin{matrix} 3^1.5.13.11.19 \\ 3^1.5.13.239 \end{matrix}$ }	VII { $\begin{matrix} 3^1.7^1.13.5.41 \\ 3^1.7^1.13.251 \end{matrix}$ }	
VIII { $\begin{matrix} 3^1.5.7.53.1889 \\ 3^1.5.7.102059 \end{matrix}$ }	IX { $\begin{matrix} 2^1.13.17.389.509 \\ 2^1.13.17.198899 \end{matrix}$ }	
X { $\begin{matrix} 3^1.5.19.37.7.887 \\ 3^1.5.19.37.7103 \end{matrix}$ }	XI { $\begin{matrix} 3^1.5.11.29.89 \\ 3^1.5.11.2699 \end{matrix}$ }	
XII { $\begin{matrix} 3^1.7^1.11.13.41.461 \\ 3^1.7^1.11.13.19403 \end{matrix}$ }	XIII { $\begin{matrix} 3^1.5.13.19.29.569 \\ 3^1.5.13.19.17099 \end{matrix}$ }	
XIV { $\begin{matrix} 3^1.7^1.13.97.5.193 \\ 3^1.7^1.13.97.1163 \end{matrix}$ }	XV { $\begin{matrix} 3^1.7.13.41.163.5.977 \\ 3^1.7.13.41.163.5867 \end{matrix}$ }	
XVI { $\begin{matrix} 2^1.17.79 \\ 2.23.59 \end{matrix}$ }	XVII { $\begin{matrix} 2^1.23.1367 \\ 2^1.53.607 \end{matrix}$ }	
XVIII { $\begin{matrix} 2^1.47.89 \\ 2^1.53.79 \end{matrix}$ }	XIX { $\begin{matrix} 2^1.23.479 \\ 2^1.89.127 \end{matrix}$ }	

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O

X

9 &

5
31

5. expof-
blemata 4
aritur, in
genere est
ad multi-
tudinem



XX { ² . 23. 467 ² . 103. 107 }	XXI { ² . 17. 5119 ² . 239. 383 }
XXII { ² . 17. 10303 ² . 167. 1103 }	XXIII { ² . 19. 1439 ² . 149. 191 }
XXIV { ² . 59. 1103 ² . 79. 827 }	XXV { ² . 37. 12671 ² . 227. 2111 }
XXVI { ² . 53. 10559 ² . 79. 7127 }	XXVII { ² . 79. 11087 ² . 383. 2309 }
XXVIII { ² . 383. 9203 ² . 1151. 3067 }	XXIX { ² . 11. 17. 263 ² . 11. 43. 107 }
XXX { ³ . 5. 7. 71 ³ . 5. 17. 31 }	XXXI { ³ . 5. 13. 29. 79 ³ . 5. 13. 11. 199 }
XXXII { ³ . 5. 13. 19. 47 ³ . 5. 13. 29. 31 }	XXXIII { ³ . 5. 13. 19. 37. 1583 ³ . 5. 13. 19. 227. 263 }
XXXIV { ³ . 7. 13. 19. 11. 220499 ³ . 7. 13. 19. 89. 29399 }	XXXV { ³ . 5. 19. 37. 47 ³ . 5. 19. 7. 227 }
XXXVI { ² . 67. 37. 2411 ² . 67. 227. 401 }	XXXVII { ³ . 5. 7. 11. 29 ³ . 5. 31. 89 }
XXXVIII { ² . 5. 23. 29. 673 ² . 5. 7. 60659 }	XXXIX { ² . 5. 7. 19. 107 ² . 5. 47. 359 }
XL { ² . 11. 163. 191 ² . 31. 11807 }	XLI { ³ . 7. 13. 23. 11. 19. 367 ³ . 7. 13. 23. 79. 1103 }
XLII { ³ . 5. 23. 11. 19. 367 ³ . 5. 23. 79. 1103 }	XLIII { ² . 11. 59. 173 ² . 57. 2609 }
XLIV { ² . 11. 23. 2543 ² . 383. 1907 }	XLV { ² . 11. 23. 1871 ² . 467. 1151 }
XLVI { ² . 11. 23. 1619 ² . 719. 647 }	XLVII { ² . 11. 29. 239 ² . 191. 449 }
	XLVIII

$$\text{XLVIII} \left\{ \begin{array}{l} 2'. 29. 47. 59 \\ 2'. 17. 4799 \end{array} \right\}$$

$$\text{L} \left\{ \begin{array}{l} 2'. 23. 47. 9767 \\ 2'. 1583. 7103 \end{array} \right\}$$

$$\text{LII} \left\{ \begin{array}{l} 3'. 7. 13. 5. 17. 1187 \\ 3'. 7. 13. 131. 971 \end{array} \right\}$$

$$\text{LIV} \left\{ \begin{array}{l} 3'. 5. 11. 59. 179 \\ 3'. 5. 17. 19. 359 \end{array} \right\}$$

$$\text{LVI} \left\{ \begin{array}{l} 3'. 7. 11. 19. 47. 7019 \\ 3'. 7. 11. 19. 389. 863 \end{array} \right\}$$

$$\text{LVIII} \left\{ \begin{array}{l} 3'. 7. 13. 19. 47. 7019 \\ 3'. 7. 13. 19. 389. 863 \end{array} \right\}$$

$$\text{XLIX} \left\{ \begin{array}{l} 2'. 17. 167. 13679 \\ 2'. 809. 51071 \end{array} \right\}$$

$$\text{LI} \left\{ \begin{array}{l} 2'. 5. 13. 1187 \\ 2'. 43. 2267 \end{array} \right\}$$

$$\text{LIII} \left\{ \begin{array}{l} 3'. 7. 13. 53. 11. 211 \\ 3'. 7. 13. 53. 2543 \end{array} \right\}$$

$$\text{LV} \left\{ \begin{array}{l} 3'. 5. 17. 23. 397 \\ 3'. 5. 7. 21491 \end{array} \right\}$$

$$\text{LVII} \left\{ \begin{array}{l} 3'. 7. 11. 19. 53. 6959 \\ 3'. 7. 11. 19. 179. 2087 \end{array} \right\}$$

$$\text{LIX} \left\{ \begin{array}{l} 3'. 7. 13. 19. 53. 6959 \\ 3'. 7. 13. 19. 179. 2087 \end{array} \right\}$$

His adijcere lubet duo paria sequentia, quæ sunt formæ diversæ a præcedentibus:

$$\text{LX} \left\{ \begin{array}{l} 2'. 19. 41 \\ 2'. 199 \end{array} \right\}$$

$$\text{LXI} \left\{ \begin{array}{l} 2'. 41. 467 \\ 2'. 19. 233 \end{array} \right\}$$

Demonstratio Gemina THEOREMATIS NEUTONIANI quo traditur relatio inter coëfficientes cujusvis æquationis algebraicæ & summas potestatum radicum ejusdem.

§. I.

Postquam æquatio algebraica tam a fractionibus quam ab irrationalitate fuerit liberata, atque ad hujusmodi formam reducta:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \dots + N = 0$$

demonstrari solet in analysi, hujusmodi æquationem tot semper habere radices, siue sint reales siue imaginariæ, quot unitates contineantur in potestatis summæ exponente n . Tum vero non minus certum est, si hujus æquationis omnes radices fuerint $\alpha, \epsilon, \gamma, \delta, \epsilon, \dots \nu$, coëfficientes terminorum æquationis $A, B, C, D, E, \&c.$ ex his radicibus ita constari, ut sit:

$$A = \text{summæ omnium radicum} = \alpha + \epsilon + \gamma + \delta + \dots + \nu$$

$$B = \text{summæ productorum ex binis} = \alpha\epsilon + \alpha\gamma + \alpha\delta + \epsilon\gamma + \&c.$$

$$C = \text{summæ productorum ex ternis} = \alpha\epsilon\gamma + \&c.$$

$$D = \text{summæ productorum ex quaternis} = \alpha\epsilon\gamma\delta + \&c.$$

$$E = \text{summæ productorum ex quinis} = \alpha\epsilon\gamma\delta\epsilon + \&c.$$

&c.

Ultimumque tandem terminum absolutum $\pm N$ esse productum ex omnibus radicibus $\alpha\epsilon\gamma\delta \dots \nu$.

§. II.



§. II. Quo jam theorema, cujus demonstrationem hic tradere constitui, facilius ac brevius enunciare queam; designet \sqrt{a} summam omnium radicum; $\sqrt{a^2}$ summam quadratorum earundem radicum; $\sqrt{a^3}$ summam cuborum radicum; $\sqrt{a^4}$ summam biquadratorum istarum radicum, & ita porro: ita ut sit:

$$\sqrt{a} = a + c + v + d + e + \dots + v$$

$$\sqrt{a^2} = a^2 + c^2 + v^2 + d^2 + e^2 + \dots + v^2$$

$$\sqrt{a^3} = a^3 + c^3 + v^3 + d^3 + e^3 + \dots + v^3$$

$$\sqrt{a^4} = a^4 + c^4 + v^4 + d^4 + e^4 + \dots + v^4$$

$$\sqrt{a^5} = a^5 + c^5 + v^5 + d^5 + e^5 + \dots + v^5$$

&c.

§. III. Hac signandi ratione exposita Neutonus affirmat, istas potestatum, quæ ex singulis radicibus formantur, summam per coefficientes æquationis A, B, C, D, E, &c. ita definiri, ut sit.

$$\sqrt{a} = A$$

$$\sqrt{a^2} = A^2 - 2B$$

$$\sqrt{a^3} = A^3 - 3B\sqrt{a} + 3C$$

$$\sqrt{a^4} = A^4 - 4B\sqrt{a^2} + 6C\sqrt{a} - 4D$$

$$\sqrt{a^5} = A^5 - 5B\sqrt{a^3} + 10C\sqrt{a^2} - 10D\sqrt{a} + 5E$$

$$\sqrt{a^6} = A^6 - 6B\sqrt{a^4} + 15C\sqrt{a^2} - 20D\sqrt{a} + 15E - 6F$$

&c.

O 3

Cujus

§. II.



Cujus theorematis demonstrationem Neutonus non solum nullam tradit, sed etiam Ipse videtur ejus veritatem ex continua illatione conclusisse. Primum enim demonstratione non eget esse, $f^a = A$: & cum sit

$$A = a^2 + c^2 + v^2 + \delta^2 + e^2 + \&c. + 2ac + 2av + 2a\delta + 2cv + 2c\delta + \&c. \text{ erit } A = f^a + 2B, \text{ ideoque } f^a = A - 2B = A - 2B: \text{ similique modo veritas sequentium formularum evinci potest; sed continuo majore opus erit labore.}$$

§. IV. Cum plures jam hujus theorematis utilissimi veritatem ostenderint, eorum demonstrationes autem regulis combinationum plerumque innitantur, quæ etiam si veræ sint, tamen ab inductione plurimum pendeant, duplicem hic afferam demonstrationem, in quarum utraque inductioni nihil tribuitur. Altera quidem ex analysi infinitorum est petita, quæ etsi nimis longe remota videatur, tamen totum negotium perfecte conficit: verum tamen cum contra eam jure objici queat, hujus theorematis veritatem evictam esse oportere, antequam ad analysin infinitorum perveniatur; alteram demonstrationem adjungam, in qua nihil assumitur, nisi quod statim ab initio in explicatione naturæ æquationum tradi solet.

Demonstratio I.

§. V. Ponatur $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \dots + N = Z$ & cum æquationis $Z = 0$ radices seu valores ipsius x sint, $a, c, v, \delta, \dots, v$, quorum numerus est $= n$, erit ex natura æquationum:

$$Z = (x - a)(x - c)(x - v)(x - \delta) \dots (x - v) \text{ et}$$

et logarithmis sumendis habebitur:

$$1Z = 1(x-a) + 1(x-c) + 1(x-y) + 1(x-d) + \dots + 1(x-v)$$

Quod si jam harum formularum differentialia capiantur erit:

$$\frac{dZ}{Z} = \frac{dx}{x-a} + \frac{dx}{x-c} + \frac{dx}{x-y} + \frac{dx}{x-d} + \dots + \frac{dx}{x-v}$$

ideoque per dx dividendo fiet:

$$\frac{dZ}{Zdx} = \frac{1}{x-a} + \frac{1}{x-c} + \frac{1}{x-y} + \frac{1}{x-d} + \dots + \frac{1}{x-v}$$

Convertantur nunc singulae hae fractiones more solito in series geometricas in finitas: ob

$$\frac{1}{x-a} = \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \frac{a^3}{x^4} + \frac{a^4}{x^5} + \frac{a^5}{x^6} + \dots$$

$$\frac{1}{x-c} = \frac{1}{x} + \frac{c}{x^2} + \frac{c^2}{x^3} + \frac{c^3}{x^4} + \frac{c^4}{x^5} + \frac{c^5}{x^6} + \dots$$

$$\frac{1}{x-y} = \frac{1}{x} + \frac{y}{x^2} + \frac{y^2}{x^3} + \frac{y^3}{x^4} + \frac{y^4}{x^5} + \frac{y^5}{x^6} + \dots$$

&c.

$$\frac{1}{x-d} = \frac{1}{x} + \frac{d}{x^2} + \frac{d^2}{x^3} + \frac{d^3}{x^4} + \frac{d^4}{x^5} + \frac{d^5}{x^6} + \dots$$

His

His igitur seriebus colligendis, signisque ante expositis f^a , f^2 , f^3 &c. introducendis invenietur, quia numerus harum serie-
rum est $= n$:

$$\frac{dZ}{Zdx} = \frac{n}{x} + \frac{1}{x^2} f^a + \frac{1}{x^3} f^2 + \frac{1}{x^4} f^3 + \frac{1}{x^5} f^4 + \&c.$$

§. VI. Cum autem statuerimus:

$$Z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \dots + N$$

erit similiter differentialibus sumendis:

$$\frac{dZ}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c.$$

hincque colligetur superior formula $\frac{dZ}{Zdx}$ ita expressa ut sit:

$$\frac{\frac{dZ}{dx}}{Z} = \frac{nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c.}{x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c.}$$

quæ igitur fractio æqualis esse debet seriei supra inventæ:

$$\frac{n}{x} + \frac{1}{x^2} f^a + \frac{1}{x^3} f^2 + \frac{1}{x^4} f^3 + \frac{1}{x^5} f^4 + \&c.$$

Quare



expositis f ,
us harum serie-

Quare si utraque expressio pro $\frac{dZ}{Zdx}$ inventa per alterius deno-

minatorem $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + \&c.$ multi-
plicetur resultabit hæc æquatio:

$$\begin{aligned} nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \&c. \\ = nx^{n-1} + x^{n-2}f^a + x^{n-3}f^a + x^{n-4}f^a + x^{n-5}f^a + \&c. \\ - nAx^{n-2} - Ax^{n-3}f^a - Ax^{n-4}f^a - Ax^{n-5}f^a - \&c. \\ + nBx^{n-3} + Bx^{n-4}f^a + Bx^{n-5}f^a + \&c. \\ - nCx^{n-4} - Cx^{n-5}f^a - \&c. \\ + nDx^{n-5} + \&c. \end{aligned}$$

§. VII. Quemadmodum jam utrinque termini primi sunt æquales, necesse est ut & secundi, tertii, quarti &c. inter se seorsim æquantur; unde sequentes nascentur æquationes:

$$\begin{aligned} - (n-1)A &= f^a - nA \\ + (n-2)B &= f^a - Af^a + nB \\ - (n-3)C &= f^a - Af^a + Bf^a - nC \\ + (n-4)D &= f^a - Af^a + Bf^a - Cf^a + nD \\ &\&c. \end{aligned}$$

harumque æquationum lex, qua progrediuntur, sponte est
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manifesta. Ex his autem obtinentur formulæ illæ ipsæ, quibus theorema Newtonianum constat; scilicet:

$$f^a = A$$

$$f^a = Af^a - 2B$$

$$f^a = Af^a - Bf^a + 3C$$

$$f^a = Af^a - Bf^a + Cf^a - 4D$$

$$f^a = Af^a - Bf^a + Cf^a - Df^a + 5E$$

Quæ est altera theorematum propositi demonstratio.

Demonstratio II.

§. VIII. Quo hujus demonstrationis vis clarius perspicatur, eam ad æquationem determinati gradus accommodabo, ita tamen ut ea intelligatur ad quosvis gradus æque patere. Sit ergo proposita æquatio quinti gradus:

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

cujus quinque radices sint $a, c, \gamma, \delta, \varepsilon$. Quia igitur quælibet radix loco x substituta æquationi satisfacit, erit:

$$a^5 - Aa^4 + Ba^3 - Ca^2 + Da - E = 0$$

$$c^5 - Ac^4 + Bc^3 - Cc^2 + Dc - E = 0$$

$$\gamma^5 - A\gamma^4 + B\gamma^3 - C\gamma^2 + D\gamma - E = 0$$

$$\delta^5 - A\delta^4 + B\delta^3 - C\delta^2 + D\delta - E = 0$$

$$\varepsilon^5 - A\varepsilon^4 + B\varepsilon^3 - C\varepsilon^2 + D\varepsilon - E = 0$$

Colligentur hæc æquationes in unam summam, & ob signa supra recepta (§. 2.) habebitur:

$$f^a = Af^a + Bf^a - Cf^a + Df^a - 5E = 0$$

$$\text{seu } f^a = Af^a - Bf^a + Cf^a - Df^a + 5E$$

§. IX.

quibus the-

§. IX. Hinc dilucide patet, si æquatio proposita fuerit
gradus cujuscunque

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \dots \pm Mx \mp N = 0$$

ubi in ultimis terminis signorum ambiguum superiora valent, si
exponens summus n fuerit numerus impar, inferiora si par; fore
pariter:

$$f^n = Af^{n-1} - Bf^{n-2} + Cf^{n-3} - \dots \mp Mf \pm nN$$

us perspicua-
modabo, ita
ere. Sit er-

si quidem per a indicetur radix quælibet istius æquationis sicque
veritas Theorematis Neutroniani jam pro uno casu est ostensa. Su-
per est igitur, ut ejusdem veritatem tam pro altioribus quam pro
inferioribus radicum potestatibus demonstremus.

§. X. Pro altioribus quidem potestatibus res pari modo
patet, si enim valores a, c, y, d, e satisfaciunt æquationi

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

satisfaciant quoque sequentibus æquationibus:

$$x^6 - Ax^5 + Bx^4 - Cx^3 + Dx^2 - Ex = 0$$

$$x^7 - Ax^6 + Bx^5 - Cx^4 + Dx^3 - Ex^2 = 0$$

$$x^8 - Ax^7 + Bx^6 - Cx^5 + Dx^4 - Ex^3 = 0$$

&c.

& ob signa

Ac propterea si in unaquaque æquatione pro x singuli valores
 a, c, y, d, e substituantur, & aggregata colligantur, erit

§. IX.

P 2

f_a

$$f^6 = Af^5 - Bf^4 + Cf^3 - Df^2 + Ef$$

$$f^7 = Af^6 - Bf^5 + Cf^4 - Df^3 + Ef^2$$

$$f^8 = Af^7 - Bf^6 + Cf^5 - Df^4 + Ef^3$$

&c.

§. XI. Si ergo α denotet radicem quamcunque hujus æquationis:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \dots + Mx + N = 0$$

erit non solum, uti jam invenimus:

$$f^n = Af^{n-1} - Bf^{n-2} + Cf^{n-3} - Df^{n-4} + \dots - Mf + N$$

sed etiã ad altiores quoque potestates progrediendo erit:

$$f^{n+1} = Af^n - Bf^{n-1} + Cf^{n-2} - Df^{n-3} + \dots - Mf^n + Nf^n$$

$$f^{n+2} = Af^{n+1} - Bf^n + Cf^{n-1} - Df^{n-2} + \dots - Mf^{n+1} + Nf^{n+1}$$

$$f^{n+3} = Af^{n+2} - Bf^{n+1} + Cf^{n+1} - Df^{n+1} + \dots - Mf^{n+2} + Nf^{n+2}$$

&c.

& in genere quidem, si ad n addatur numerus quicunque m , erit

$$f^{n+m} = Af^{n+m-1} - Bf^{n+m-2} + Cf^{n+m-3} - \dots - Mf^{n+m-1} + Nf^{n+m}$$

Ubi quidem notandum est, si sit $m = 0$, ob singulas potestates

$$f^0 = 1, f^1 = f, f^2 = f^2: \&c. \text{ numerumque harum litterarum } f^m \text{ fore}$$



fore $f/a = n$, quo casu formula primo inventa in hac expressione continetur.

§. XII. Quamquam autem hæc expressio æque veritati est consentanea, si pro m accipiatur numerus negativus: hincque pro æquatione quinti gradus assumta

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

sequentes formulæ pariter locum habent:

$$fa^4 = Afa^3 - Bfa^2 + Cfa^1 - Dfa^0 + Efa^{-1}$$

$$fa^3 = Afa^2 - Bfa^1 + Cfa^0 - Dfa^{-1} + Efa^{-2}$$

$$fa^2 = Afa^1 - Bfa^0 + Cfa^{-1} - Dfa^{-2} + Efa^{-3}$$

$$fa^1 = Afa^0 - Bfa^{-1} + Cfa^{-2} - Dfa^{-3} + Efa^{-4}$$

$$fa^0 = Afa^{-1} - Bfa^{-2} + Cfa^{-3} - Dfa^{-4} + Efa^{-5}$$

$$fa^{-1} = Afa^{-2} - Bfa^{-3} + Cfa^{-4} - Dfa^{-5} + Efa^{-6}$$

$$fa^{-2} = Afa^{-3} - Bfa^{-4} + Cfa^{-5} - Dfa^{-6} + Efa^{-7}$$

$$fa^{-3} = Afa^{-4} - Bfa^{-5} + Cfa^{-6} - Dfa^{-7} + Efa^{-8}$$

$$fa^{-4} = Afa^{-5} - Bfa^{-6} + Cfa^{-7} - Dfa^{-8} + Efa^{-9}$$

$$fa^{-5} = Afa^{-6} - Bfa^{-7} + Cfa^{-8} - Dfa^{-9} + Efa^{-10}$$

$$fa^{-6} = Afa^{-7} - Bfa^{-8} + Cfa^{-9} - Dfa^{-10} + Efa^{-11}$$

$$fa^{-7} = Afa^{-8} - Bfa^{-9} + Cfa^{-10} - Dfa^{-11} + Efa^{-12}$$

$$fa^{-8} = Afa^{-9} - Bfa^{-10} + Cfa^{-11} - Dfa^{-12} + Efa^{-13}$$

$$fa^{-9} = Afa^{-10} - Bfa^{-11} + Cfa^{-12} - Dfa^{-13} + Efa^{-14}$$

$$fa^{-10} = Afa^{-11} - Bfa^{-12} + Cfa^{-13} - Dfa^{-14} + Efa^{-15}$$

$$fa^{-11} = Afa^{-12} - Bfa^{-13} + Cfa^{-14} - Dfa^{-15} + Efa^{-16}$$

$$fa^{-12} = Afa^{-13} - Bfa^{-14} + Cfa^{-15} - Dfa^{-16} + Efa^{-17}$$

$$fa^{-13} = Afa^{-14} - Bfa^{-15} + Cfa^{-16} - Dfa^{-17} + Efa^{-18}$$

$$fa^{-14} = Afa^{-15} - Bfa^{-16} + Cfa^{-17} - Dfa^{-18} + Efa^{-19}$$

$$fa^{-15} = Afa^{-16} - Bfa^{-17} + Cfa^{-18} - Dfa^{-19} + Efa^{-20}$$

$$fa^{-16} = Afa^{-17} - Bfa^{-18} + Cfa^{-19} - Dfa^{-20} + Efa^{-21}$$

$$fa^{-17} = Afa^{-18} - Bfa^{-19} + Cfa^{-20} - Dfa^{-21} + Efa^{-22}$$

$$fa^{-18} = Afa^{-19} - Bfa^{-20} + Cfa^{-21} - Dfa^{-22} + Efa^{-23}$$

$$fa^{-19} = Afa^{-20} - Bfa^{-21} + Cfa^{-22} - Dfa^{-23} + Efa^{-24}$$

$$fa^{-20} = Afa^{-21} - Bfa^{-22} + Cfa^{-23} - Dfa^{-24} + Efa^{-25}$$

$$fa^{-21} = Afa^{-22} - Bfa^{-23} + Cfa^{-24} - Dfa^{-25} + Efa^{-26}$$

$$fa^{-22} = Afa^{-23} - Bfa^{-24} + Cfa^{-25} - Dfa^{-26} + Efa^{-27}$$

$$fa^{-23} = Afa^{-24} - Bfa^{-25} + Cfa^{-26} - Dfa^{-27} + Efa^{-28}$$

$$fa^{-24} = Afa^{-25} - Bfa^{-26} + Cfa^{-27} - Dfa^{-28} + Efa^{-29}$$

$$fa^{-25} = Afa^{-26} - Bfa^{-27} + Cfa^{-28} - Dfa^{-29} + Efa^{-30}$$

$$fa^{-26} = Afa^{-27} - Bfa^{-28} + Cfa^{-29} - Dfa^{-30} + Efa^{-31}$$

$$fa^{-27} = Afa^{-28} - Bfa^{-29} + Cfa^{-30} - Dfa^{-31} + Efa^{-32}$$

$$fa^{-28} = Afa^{-29} - Bfa^{-30} + Cfa^{-31} - Dfa^{-32} + Efa^{-33}$$

$$fa^{-29} = Afa^{-30} - Bfa^{-31} + Cfa^{-32} - Dfa^{-33} + Efa^{-34}$$

$$fa^{-30} = Afa^{-31} - Bfa^{-32} + Cfa^{-33} - Dfa^{-34} + Efa^{-35}$$

$$fa^{-31} = Afa^{-32} - Bfa^{-33} + Cfa^{-34} - Dfa^{-35} + Efa^{-36}$$

$$fa^{-32} = Afa^{-33} - Bfa^{-34} + Cfa^{-35} - Dfa^{-36} + Efa^{-37}$$

$$fa^{-33} = Afa^{-34} - Bfa^{-35} + Cfa^{-36} - Dfa^{-37} + Efa^{-38}$$

$$fa^{-34} = Afa^{-35} - Bfa^{-36} + Cfa^{-37} - Dfa^{-38} + Efa^{-39}$$

$$fa^{-35} = Afa^{-36} - Bfa^{-37} + Cfa^{-38} - Dfa^{-39} + Efa^{-40}$$

$$fa^{-36} = Afa^{-37} - Bfa^{-38} + Cfa^{-39} - Dfa^{-40} + Efa^{-41}$$

$$fa^{-37} = Afa^{-38} - Bfa^{-39} + Cfa^{-40} - Dfa^{-41} + Efa^{-42}$$

$$fa^{-38} = Afa^{-39} - Bfa^{-40} + Cfa^{-41} - Dfa^{-42} + Efa^{-43}$$

$$fa^{-39} = Afa^{-40} - Bfa^{-41} + Cfa^{-42} - Dfa^{-43} + Efa^{-44}$$

$$fa^{-40} = Afa^{-41} - Bfa^{-42} + Cfa^{-43} - Dfa^{-44} + Efa^{-45}$$

$$fa^{-41} = Afa^{-42} - Bfa^{-43} + Cfa^{-44} - Dfa^{-45} + Efa^{-46}$$

$$fa^{-42} = Afa^{-43} - Bfa^{-44} + Cfa^{-45} - Dfa^{-46} + Efa^{-47}$$

$$fa^{-43} = Afa^{-44} - Bfa^{-45} + Cfa^{-46} - Dfa^{-47} + Efa^{-48}$$

$$fa^{-44} = Afa^{-45} - Bfa^{-46} + Cfa^{-47} - Dfa^{-48} + Efa^{-49}$$

$$fa^{-45} = Afa^{-46} - Bfa^{-47} + Cfa^{-48} - Dfa^{-49} + Efa^{-50}$$

$$fa^{-46} = Afa^{-47} - Bfa^{-48} + Cfa^{-49} - Dfa^{-50} + Efa^{-51}$$

$$fa^{-47} = Afa^{-48} - Bfa^{-49} + Cfa^{-50} - Dfa^{-51} + Efa^{-52}$$

$$fa^{-48} = Afa^{-49} - Bfa^{-50} + Cfa^{-51} - Dfa^{-52} + Efa^{-53}$$

$$fa^{-49} = Afa^{-50} - Bfa^{-51} + Cfa^{-52} - Dfa^{-53} + Efa^{-54}$$

$$fa^{-50} = Afa^{-51} - Bfa^{-52} + Cfa^{-53} - Dfa^{-54} + Efa^{-55}$$

$$fa^{-51} = Afa^{-52} - Bfa^{-53} + Cfa^{-54} - Dfa^{-55} + Efa^{-56}$$

$$fa^{-52} = Afa^{-53} - Bfa^{-54} + Cfa^{-55} - Dfa^{-56} + Efa^{-57}$$

$$fa^{-53} = Afa^{-54} - Bfa^{-55} + Cfa^{-56} - Dfa^{-57} + Efa^{-58}$$

$$fa^{-54} = Afa^{-55} - Bfa^{-56} + Cfa^{-57} - Dfa^{-58} + Efa^{-59}$$

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$$fa^{-56} = Afa^{-57} - Bfa^{-58} + Cfa^{-59} - Dfa^{-60} + Efa^{-61}$$

$$fa^{-57} = Afa^{-58} - Bfa^{-59} + Cfa^{-60} - Dfa^{-61} + Efa^{-62}$$

$$fa^{-58} = Afa^{-59} - Bfa^{-60} + Cfa^{-61} - Dfa^{-62} + Efa^{-63}$$

$$fa^{-59} = Afa^{-60} - Bfa^{-61} + Cfa^{-62} - Dfa^{-63} + Efa^{-64}$$

$$fa^{-60} = Afa^{-61} - Bfa^{-62} + Cfa^{-63} - Dfa^{-64} + Efa^{-65}$$

$$fa^{-61} = Afa^{-62} - Bfa^{-63} + Cfa^{-64} - Dfa^{-65} + Efa^{-66}$$

$$fa^{-62} = Afa^{-63} - Bfa^{-64} + Cfa^{-65} - Dfa^{-66} + Efa^{-67}$$

$$fa^{-63} = Afa^{-64} - Bfa^{-65} + Cfa^{-66} - Dfa^{-67} + Efa^{-68}$$

$$fa^{-64} = Afa^{-65} - Bfa^{-66} + Cfa^{-67} - Dfa^{-68} + Efa^{-69}$$

$$fa^{-65} = Afa^{-66} - Bfa^{-67} + Cfa^{-68} - Dfa^{-69} + Efa^{-70}$$

$$fa^{-66} = Afa^{-67} - Bfa^{-68} + Cfa^{-69} - Dfa^{-70} + Efa^{-71}$$

$$fa^{-67} = Afa^{-68} - Bfa^{-69} + Cfa^{-70} - Dfa^{-71} + Efa^{-72}$$

$$fa^{-68} = Afa^{-69} - Bfa^{-70} + Cfa^{-71} - Dfa^{-72} + Efa^{-73}$$

$$fa^{-69} = Afa^{-70} - Bfa^{-71} + Cfa^{-72} - Dfa^{-73} + Efa^{-74}$$

$$fa^{-70} = Afa^{-71} - Bfa^{-72} + Cfa^{-73} - Dfa^{-74} + Efa^{-75}$$

$$fa^{-71} = Afa^{-72} - Bfa^{-73} + Cfa^{-74} - Dfa^{-75} + Efa^{-76}$$

$$fa^{-72} = Afa^{-73} - Bfa^{-74} + Cfa^{-75} - Dfa^{-76} + Efa^{-77}$$

$$fa^{-73} = Afa^{-74} - Bfa^{-75} + Cfa^{-76} - Dfa^{-77} + Efa^{-78}$$

$$fa^{-74} = Afa^{-75} - Bfa^{-76} + Cfa^{-77} - Dfa^{-78} + Efa^{-79}$$

$$fa^{-75} = Afa^{-76} - Bfa^{-77} + Cfa^{-78} - Dfa^{-79} + Efa^{-80}$$

$$fa^{-76} = Afa^{-77} - Bfa^{-78} + Cfa^{-79} - Dfa^{-80} + Efa^{-81}$$

$$fa^{-77} = Afa^{-78} - Bfa^{-79} + Cfa^{-80} - Dfa^{-81} + Efa^{-82}$$

$$fa^{-78} = Afa^{-79} - Bfa^{-80} + Cfa^{-81} - Dfa^{-82} + Efa^{-83}$$

$$fa^{-79} = Afa^{-80} - Bfa^{-81} + Cfa^{-82} - Dfa^{-83} + Efa^{-84}$$

$$fa^{-80} = Afa^{-81} - Bfa^{-82} + Cfa^{-83} - Dfa^{-84} + Efa^{-85}$$

$$fa^{-81} = Afa^{-82} - Bfa^{-83} + Cfa^{-84} - Dfa^{-85} + Efa^{-86}$$

$$fa^{-82} = Afa^{-83} - Bfa^{-84} + Cfa^{-85} - Dfa^{-86} + Efa^{-87}$$

$$fa^{-83} = Afa^{-84} - Bfa^{-85} + Cfa^{-86} - Dfa^{-87} + Efa^{-88}$$

$$fa^{-84} = Afa^{-85} - Bfa^{-86} + Cfa^{-87} - Dfa^{-88} + Efa^{-89}$$

$$fa^{-85} = Afa^{-86} - Bfa^{-87} + Cfa^{-88} - Dfa^{-89} + Efa^{-90}$$

$$fa^{-86} = Afa^{-87} - Bfa^{-88} + Cfa^{-89} - Dfa^{-90} + Efa^{-91}$$

$$fa^{-87} = Afa^{-88} - Bfa^{-89} + Cfa^{-90} - Dfa^{-91} + Efa^{-92}$$

$$fa^{-88} = Afa^{-89} - Bfa^{-90} + Cfa^{-91} - Dfa^{-92} + Efa^{-93}$$

$$fa^{-89} = Afa^{-90} - Bfa^{-91} + Cfa^{-92} - Dfa^{-93} + Efa^{-94}$$

$$fa^{-90} = Afa^{-91} - Bfa^{-92} + Cfa^{-93} - Dfa^{-94} + Efa^{-95}$$

$$fa^{-91} = Afa^{-92} - Bfa^{-93} + Cfa^{-94} - Dfa^{-95} + Efa^{-96}$$

$$fa^{-92} = Afa^{-93} - Bfa^{-94} + Cfa^{-95} - Dfa^{-96} + Efa^{-97}$$

$$fa^{-93} = Afa^{-94} - Bfa^{-95} + Cfa^{-96} - Dfa^{-97} + Efa^{-98}$$

$$fa^{-94} = Afa^{-95} - Bfa^{-96} + Cfa^{-97} - Dfa^{-98} + Efa^{-99}$$

$$fa^{-95} = Afa^{-96} - Bfa^{-97} + Cfa^{-98} - Dfa^{-99} + Efa^{-100}$$

$$fa^{-96} = Afa^{-97} - Bfa^{-98} + Cfa^{-99} - Dfa^{-100} + Efa^{-101}$$

$$fa^{-97} = Afa^{-98} - Bfa^{-99} + Cfa^{-100} - Dfa^{-101} + Efa^{-102}$$

$$fa^{-98} = Afa^{-99} - Bfa^{-100} + Cfa^{-101} - Dfa^{-102} + Efa^{-103}$$

$$fa^{-99} = Afa^{-100} - Bfa^{-101} + Cfa^{-102} - Dfa^{-103} + Efa^{-104}$$

$$fa^{-100} = Afa^{-101} - Bfa^{-102} + Cfa^{-103} - Dfa^{-104} + Efa^{-105}$$

$$fa^{-101} = Afa^{-102} - Bfa^{-103} + Cfa^{-104} - Dfa^{-105} + Efa^{-106}$$

$$fa^{-102} = Afa^{-103} - Bfa^{-104} + Cfa^{-105} - Dfa^{-106} + Efa^{-107}$$

$$fa^{-103} = Afa^{-104} - Bfa^{-105} + Cfa^{-106} - Dfa^{-107} + Efa^{-108}$$

$$fa^{-104} = Afa^{-105} - Bfa^{-106} + Cfa^{-107} - Dfa^{-108} + Efa^{-109}$$

$$fa^{-105} = Afa^{-106} - Bfa^{-107} + Cfa^{-108} - Dfa^{-109} + Efa^{-110}$$

$$fa^{-106} = Afa^{-107} - Bfa^{-108} + Cfa^{-109} - Dfa^{-110} + Efa^{-111}$$

$$fa^{-107} = Afa^{-108} - Bfa^{-109} + Cfa^{-110} - Dfa^{-111} + Efa^{-112}$$

$$fa^{-108} = Afa^{-109} - Bfa^{-110} + Cfa^{-111} - Dfa^{-112} + Efa^{-113}$$

$$fa^{-109} = Afa^{-110} - Bfa^{-111} + Cfa^{-112} - Dfa^{-113} + Efa^{-114}$$

$$fa^{-110} = Afa^{-111} - Bfa^{-112} + Cfa^{-113} - Dfa^{-114} + Efa^{-115}$$

$$fa^{-111} = Afa^{-112} - Bfa^{-113} + Cfa^{-114} - Dfa^{-115} + Efa^{-116}$$

$$fa^{-112} = Afa^{-113} - Bfa^{-114} + Cfa^{-115} - Dfa^{-116} + Efa^{-117}$$

$$fa^{-113} = Afa^{-114} - Bfa^{-115} + Cfa^{-116} - Dfa^{-117} + Efa^{-118}$$

$$fa^{-114} = Afa^{-115} - Bfa^{-116} + Cfa^{-117} - Dfa^{-118} + Efa^{-119}$$

$$fa^{-115} = Afa^{-116} - Bfa^{-117} + Cfa^{-118} - Dfa^{-119} + Efa^{-120}$$

$$fa^{-116} = Afa^{-117} - Bfa^{-118} + Cfa^{-119} - Dfa^{-120} + Efa^{-121}$$

$$fa^{-117} = Afa^{-118} - Bfa^{-119} + Cfa^{-120} - Dfa^{-121} + Efa^{-122}$$

$$fa^{-118} = Afa^{-119} - Bfa^{-120} + Cfa^{-121} - Dfa^{-122} + Efa^{-123}$$

$$fa^{-119} = Afa^{-120} - Bfa^{-121} + Cfa^{-122} - Dfa^{-123} + Efa^{-124}$$

$$fa^{-120} = Afa^{-121} - Bfa^{-122} + Cfa^{-123} - Dfa^{-124} + Efa^{-125}$$

$$fa^{-121} = Afa^{-122} - Bfa^{-123} + Cfa^{-124} - Dfa^{-125} + Efa^{-126}$$

$$fa^{-122} = Afa^{-123} - Bfa^{-124} + Cfa^{-125} - Dfa^{-126} + Efa^{-127}$$

$$fa^{-123} = Afa^{-124} - Bfa^{-125} + Cfa^{-126} - Dfa^{-127} + Efa^{-128}$$

$$fa^{-124} = Afa^{-125} - Bfa^{-126} + Cfa^{-127} - Dfa^{-128} + Efa^{-129}$$

$$fa^{-125} = Afa^{-126} - Bfa^{-127} + Cfa^{-128} - Dfa^{-129} + Efa^{-130}$$

$$fa^{-126} = Afa^{-127} - Bfa^{-128} + Cfa^{-129} - Dfa^{-130} + Efa^{-131}$$

$$fa^{-127} = Afa^{-128} - Bfa^{-129} + Cfa^{-130} - Dfa^{-131} + Efa^{-132}$$

$$fa^{-128} = Afa^{-129} - Bfa^{-130} + Cfa^{-131} - Dfa^{-132} + Efa^{-133}$$

$$fa^{-129} = Afa^{-130} - Bfa^{-131} + Cfa^{-132} - Dfa^{-133} + Efa^{-134}$$

$$fa^{-130} = Afa^{-131} - Bfa^{-132} + Cfa^{-133} - Dfa^{-134} + Efa^{-135}$$

$$fa^{-131} = Afa^{-132} - Bfa^{-133} + Cfa^{-134} - Dfa^{-135} + Efa^{-136}$$

$$fa^{-132} = Afa^{-133} - Bfa^{-134} + Cfa^{-135} - Dfa^{-136} + Efa^{-137}$$

$$fa^{-133} = Afa^{-134} - Bfa^{-135} + Cfa^{-136} - Dfa^{-137} + Efa^{-138}$$

$$fa^{-134} = Afa^{-135} - Bfa^{-136} + Cfa^{-137} - Dfa^{-138} + Efa^{-139}$$

$$fa^{-135} = Afa^{-136} - Bfa^{-137} + Cfa^{-138} - Dfa^{-139} + Efa^{-140}$$



Formentur retinendis iisdem coefficientibus sequentes æquationes inferiorum graduum:

I. $x - A = 0$. Radix sit p

II. $x^2 - Ax + B = 0$. Radix quælibet sit q

III. $x^3 - Ax^2 + Bx - C = 0$. Sit radix quælibet r

IV. $x^4 - Ax^3 + Bx^2 - Cx + D = 0$. Radix quælibet s

Quarum æquationum radices, etiamsi inter se maxime discrepent, tamen in his singulis æquationibus eandem constituent summam $= A$. Deinde remota prima summa productorum ex binis radicibus ubique erit eadem $= B$: Tum summa productorum ex ternis radicibus ubique erit $= C$, præter æquationes scilicet I & II, ubi C non occurrit. Similiter in IV & propofita summa productorum ex quaternis radicibus erit eadem $= D$.

§. XIV. In quibus autem æquationibus non solum summa radicum est eadem, sed etiam summa productorum ex binis radicibus, ibi quoque summa quadratorum radicum est eadem. Sin autem præterea summa productorum ex ternis radicibus fuerit eadem, tum summa quoque cuborum omnium radicum erit eadem. Atque si insuper summa productorum ex quaternis radicibus fuerit eadem, tum quoque summa biquadratorum omnium radicum erit eadem atque ita porro. Hic scilicet assumo, quod facile concedetur, summam quadratorum per summam radicum & summam productorum ex binis determinari; summam cuborum autem præterea requirere summam factorum ex ternis radicibus; ac summam biquadratorum præterea summam factorum ex quaternis

ternis radicibus, & ita porro; quod quidem demonstratu non esset difficile.

§. XV. In æquationibus ergo inferiorum graduum, quarum radices denotantur respectivè per litteras p, q, r, s , dum ipsius propositæ quinti gradus quælibet radix littera a indicatur, erit:

$$p^5 = p^4 = p^3 = p^2 = p$$

$$q^5 = q^4 = q^3 = q^2 = q$$

$$r^5 = r^4 = r^3 = r^2 = r$$

$$s^5 = s^4 = s^3 = s^2 = s$$

At per ea quæ ante §. 9 demonstravimus est

$$p^5 = A$$

$$q^5 = Aq - 2B$$

$$r^5 = Ar^3 - Br^2 + 3C$$

$$s^5 = As^3 - Bs^2 + Cs - 4D$$

Hinc ergo nanciscimur pro æquatione quinti gradus proposita: $x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$ has formulas

sa

$$f^{\circ} = A$$

$$f^{\circ 2} = Af^{\circ} - 2B$$

$$f^{\circ 3} = Af^{\circ 2} - Bf^{\circ} + 3C$$

$$f^{\circ 4} = Af^{\circ 3} - Bf^{\circ 2} + Cf^{\circ} - 4D$$

§. XVI. In æquatione ergo cujuscunque gradus propo-
fita:

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = N = 0$$

si quælibet radix littera α indicetur erit:

$$f^{\circ} = A$$

$$f^{\circ 2} = Af^{\circ} - 2B$$

$$f^{\circ 3} = Af^{\circ 2} - Bf^{\circ} + 3C$$

$$f^{\circ 4} = Af^{\circ 3} - Bf^{\circ 2} + Cf^{\circ} - 4D$$

$$f^{\circ 5} = Af^{\circ 4} - Bf^{\circ 3} + Cf^{\circ 2} - Df^{\circ} + 5E$$

&c.

Hocque modo veritas Theorematis Newtoniani pariter ha-
betur demonstrata.

Ani-



Animadversiones

in

Rectificationem Ellipsis.

§. I.

Tab. I.

Ellipsis rectificatio tot jam variis methodis est frustra tentata, ut non solum comparationem arcuum ellipticorum cum lineis rectis, sed etiam ne cum circularibus quidem aut parabolicis expectare nequeamus. Cum enim formula illa differentialis, cujus integrale arcum ellipticum indefinitum exprimit, nullo modo ab irrationalitate liberari queat; certum hoc est signum, ejus integrationem non solum non algebraice, sed etiam ne concessis quidem circuli & hyperbolæ quadraturis perfici posse. Quod cum tenendum sit de rectificatione ellipsis indefinita, hinc adhuc minime sequitur, arcum quempiam definitum veluti totam perimetrum ellipsis omnem comparationem cum lineis vel rectis vel circularibus penitus respuere: propterea quod jam innumerabiles curvæ assignari possunt, indefinite æque parum rectificabiles acque ellipsis, in quibus tamen arcus definiti per lineas rectas mensurari queant.

§. II. Missa igitur rectificatione ellipsis indefinita, definitam potius sum aggressus; experturus, utrum tota cujusque ellipsis perimenter non commode possit ad mensuras cognititas, quorum etiam logarithmos & arcus circulares refero, per expressiones finitas revocari. Quamquam autem in hac investigatione nihil admodum sum consecutus, quod scopo meo satisfecisset; tamen præter expectationem nonnulla se mihi obtulerunt phaenomena.

Euleri Opuscula Tom. II. Q fatis



fatis singularia, quibus theoria linearum curvarum non mediocriter promoveri videtur. Tum vero etiam difficultates, quæ in toto hoc calculo occurrerunt, ansam mihi præbuerunt quædam insignia artificia inveniendi, quæ tam in calculo integrali, quam in theoria serierum infinitarum ingentem utilitatem sæpius asferre posse videntur. Quamobrem operæ pretium fore existimavi, si has speculationes totumque quasi filum calculorum meorum dilucide exposuero.

Propositio.

Fig. 1. §. III. Super data recta AC tanquam altero semiaxe descriptos concipio infinitos quadrantes ellipticos AP, AB, Ap, quorum ergo omnium est centrum C, alteri vero semiaxes conjugati sunt CP, CB, Cp. Tum ex singulis punctis P, B, p arcus elliptici PA, BA, pA in directum extendantur, ita ut quælibet PQ sit recta CA parallela & quadranti elliptico PTA æqualis: quod si ubique fieri concipiatur, puncta hæc Q sita erunt in linea quadam curva AQDp, cujus naturam investigare constitui.

Ad genesis hujus curvæ vel leviter attendenti mox patebit, eam sequentes habere proprietates; quas evolvam, antequam in ipsam hujus curvæ indolem diligentius inquiram; ut ejus figura & ductus saltem obiter perspiciatur.

§. IV. Primum igitur si in recta indefinita CBp quæ ad datam CA est normalis, capiatur quævis abscissa CP, applicata PC, quæ ei respondet, erit æqualis quadranti perimetri ellipsis; cujus semiaxes conjugati sunt, recta data CA & ipsa abscissa CP. Hinc si capiatur abscissa CB = CA, quo casu quadrans ellipticus abibi in quadrantem circularem AB, applicata respondens BD æqualis erit quartæ parti peripheriæ circuli radio AC descripti. Unde si ratio diametri ad peripheriam ponatur = 1:π, erit ista applicata

Q

BD



non mediocri-
leates, quæ in
crunt quædam
integrali, quan-
n sæpius asserre
e existimavi, si
meorum dila-

$BD = \frac{1}{2} \pi$. AC: five ob $\pi = 3$, 1415926535897932 erit $BD = 1,5707963267948961$. AC.

§. V. Secundo: Si abscissa CP evanescat, ellipsis evadet infinite angusta, atque cum linea recta confundetur. Hoc ergo casu quadrans ellipticus abibit in ipsam lineam AC, cui propterea applicata abscissæ evanescenti respondens erit æqualis. Quare ipsa recta CA erit applicata puncto C respondens, & curva quæstita per punctum A transibit. Hujus ergo curvæ jam duo habemus puncta cognita A & D, quorum alterum A geometricæ datur, alterum vero D per rationem diametri ad peripheriam definitur.

et scriptor con-
struam ergo omni-
CP, CB, Cp.
§. p. A in directum
& quadranti il-
luminato hæc Q sit
investigare comp-
i mox patebit,
, antequam in
ut ejus figura

§. VI. Tertio: Ex cognito quovis curvæ puncto Q intra A & D sito, semper aliud quoddam curvæ punctum q ultra D situm definiri potest. Capiatur enim Cp tertia proportionalis ad CP & CA, ut sit $Cp = \frac{CA \cdot CA}{CP}$, quia est $CP : CA = CA : Cp$, erit quadrans ellipticus Ap similis quadranti elliptico AP, cum utrinque eadem sit ratio inter semiaxes conjugatos. Hinc erit arcus Ap ad arcum AP ut AC ad CP, ideoque $pq : PQ = AC : CP$ seu $pq = \frac{AC \cdot PQ}{CP}$. Consequenter si curvæ quæstitæ arcus AD tantum jam fuerit descriptus, ex eo reliqua curvæ pars Dq in infinitum extensa definitur.

Bp quæ ad da-
applicata PC,
ri ellipsis, cu-
illa CP. Hinc
lipticus abibit
BD æqualis
ti. Unde si
illa applicata
BD

§. VII. Quarto: Hinc jam insignis proprietas æquationis, qua natura curvæ AQDq exprimitur, agnoscitur. Si enim recta data AC unitate designetur, ut sit $AC = 1$; abscissa autem quævis unitate minor $CP = p$, eique respondens applicata $PQ = q$; tum vero ponatur abscissa illa altera $Cp = P$ & applicata $pq = Q$; erit $P = \frac{1}{p}$ & $Q = \frac{q}{p}$. Quare cum inter P & Q eadem esse

-011

Q 2

esse



esse debeat æquatio quæ est inter p & q , patet æquationem inter p & q nullam mutationem esse subituram, si in ea loco p ubique scribatur $\frac{1}{p}$ & $\frac{q}{p}$ loco q . Unde qualis ipsius p functio sit q conjicere licet.

§. VIII. Quinto: Patet crescentibus abscissis CP applicatas continuo crescere, cum semper sint majores quam abscissæ. Verum si abscissæ statuantur infinitæ, applicatæ ipsis fient æquales: discrimen enim prodibit infinite parvum; unde colligimus quæsitam curvam habere asymptotam, & quidem rectam CV angulum rectum ACB bisecantem. Forma igitur hujus curvæ similis erit hyperbolæ æquilateræ centrum in C , axem CA & asymptotam CV habentis. Ex descriptione porro intelligitur, curvam infra rectam CA productam sui similem fore, ideoque rectam CA ejus fore diametrum perinde atque hyperbolæ. Veruntamen, hoc facile perspicitur, nostram curvam multo lentius ad asymptotam suam CV appropinquare quam hyperbolam. Nam in hyperbola æquilatera, cui nostram curvam comparamus, quævis applicata PQ æqualis est rectæ lineæ AP ; unde cum applicata nostræ curvæ arcui AP sit æqualis, patet hyperbolam nostræ curvæ fore circum scriptam, ita tamen ut in initio A , & in spatio infinito se mutuo tangant.

§. IX. His affectionibus latius patentibus in genere notatis, in ipsam hujus curvæ naturam accuratius inquiramus, ac proposita quacunq; abscissa $CP = p$; valorem respondentis applicatæ $PQ = q$ investigemus; qui cum expressione finita contineri nequeat, per seriem infinitam exhiberi debebit. Sequens igitur resolutio oportet

Pro-



Problema.

Ex datis semiaxibus CA & CP quadrantis elliptici CAP per seriem infinitam definire longitudinem arcus quadrantis ATP.

Solutio.

§. X. Cum vocatus sit alter semiaxis $AC = 1$, alter vero, $CP = p$, & arcus $AYP = q$, quærat^{ur} primo arcus quivis indefinitus PY , qui vocetur s . Jam ducta ad CP applicata normali YX , sit $CX = x$ & $XY = y$, erit ex natura ellipsis $x = p \sqrt{(1 - yy)}$, hincque $dx = \frac{-pydy}{\sqrt{(1 - yy)}}$. Fieri ergo ob $ds = \sqrt{(dx^2 + dy^2)}$

$$ds = \frac{dy \sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}}$$

$$\text{unde integrando erit arcus } s = \int \frac{dy \sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}}$$

quæ integratio ita institui debet, ut posito $y = 0$ fiat quoque $s = 0$, quia evanescente applicata $XY = y$ simul $PY = s$ evanescit. Hoc igitur integrali invento si ponatur $y = CA = 1$, arcus indefinitus PY abit in longitudinem quadrantis elliptici $PYA = q$, quem quærimus, ita ut sit

$$q = \int \frac{dy \sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}}$$

siquidem peracta integratione ponatur $y = 1$.

§. XI. Ad institutum ergo nostrum non est necesse, ut quæramus valorem integralis hujus indefiniti, sed eum tantum, quem induit, si post integrationem variabili y tribuatur valor determinatus $= 1$: quo pacto series multo simplicior valorem q exprimens obtineri poterit. Ponatur enim brevitatis gratia

Q 3

1-PP



$1 - pp = nn$, ut sit $\sqrt{1 - yy + ppyy} = \sqrt{1 - nnyy}$
eritque hanc formulam in seriem evolvendo.

$$\sqrt{1 - nnyy} = 1 - \frac{1}{2} nnyy - \frac{1 \cdot 1}{2 \cdot 4} n^4 y^4 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} n^6 y^6 - \&c.$$

Quo valore substituto pro $\sqrt{1 - yy + ppyy}$, arcus q ita exprimitur ut sit:

$$q = \int \frac{dy}{\sqrt{1 - yy}} - \frac{1}{2} nn \int \frac{yy dy}{\sqrt{1 - yy}} - \frac{1 \cdot 1}{2 \cdot 4} n^4 \int \frac{y^4 dy}{\sqrt{1 - yy}} \\ - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \int \frac{y^6 dy}{\sqrt{1 - yy}} \&c.$$

si quidem in singulis his integralibus post integrationem ponatur $y = 1$.

§. XII. Evolvamus ergo singula hæc integralia; ac primo quidem ex circulo manifestum est, formulam $\int \frac{dy}{\sqrt{1 - yy}}$ exprimere arcum circuli cujus sinus $= y$ pro radio $= 1$: unde posito $y = 1$, hæc formula dabit quartam peripheriæ partem, cujus radius $= 1$. Ideoque posita ratione diametri ad peripheriam $= 1 : \pi$, erit $\int \frac{dy}{\sqrt{1 - yy}} = \frac{\pi}{2}$; sicque jam adepti sumus valorem primi termini in serie nostra ante inventa.

§. XIII. Reliqui termini pari modo per valorem π commode poterunt exprimi; cujusvis enim termini integratio ad integrationem præcedentis reducitur: quod quo facilius intelligatur

consideremus formulam quamcunque $\int \frac{y^m dy}{\sqrt{1 - yy}}$; erit sequens

$\int y$



—nyy)

1.1.3
2.4.6
y · d.c.
y), arcus q ita

$$\int \frac{y^{\mu} dy}{V(1-yy)}$$

tionem ponatur

gralia; ac pri-
dy
(1-yy) ex-

= 1: unde po-
re partem, cu-
ad peripheriam

pti sumus valo-

alorem = con-
tegratio ad in-
us intelligatur

; erit sequens
ff

$$\int \frac{y^{\mu+2} dy}{V(1-yy)}$$

Jam assumamus hanc formulam algebraicam

$$y^{\mu+1} V(1-yy), \text{ cujus differentiale cum sit} =$$

$$\frac{(\mu+1)y^{\mu} dy - (\mu+2)y^{\mu+2} dy}{V(1-yy)} \text{ erit vicissim}$$

$$(\mu+1) \int \frac{y^{\mu} dy}{V(1-yy)} - (\mu+2) \int \frac{y^{\mu+2} dy}{V(1-yy)} = y^{\mu+1} V(1-yy)$$

unde colligimus fore

$$\int \frac{y^{\mu+2} dy}{V(1-yy)} = \frac{\mu+1}{\mu+2} \int \frac{y^{\mu} dy}{V(1-yy)} - \frac{1}{\mu+2} y^{\mu+1} V(1-yy)$$

Quare invento integrali $\int \frac{y^{\mu} dy}{V(1-yy)}$ ex eo facile elicitur integra-

$$\text{le sequens } \int \frac{y^{\mu+2} dy}{V(1-yy)}$$

§. XIV. Quoniam vero eos tantum horum integralium valores desideramus, qui prodeunt posito $y = 1$; hoc casu quan-

titas algebraica $\frac{1}{\mu+1} y^{\mu+1} V(1-yy)$ evanescit, eritque ge-
neratim pro casu $y = 1$

fy

$$\int \frac{y^{\mu+2} dy}{V(1-yy)} = \frac{\mu+1}{\mu+2} \int \frac{y^{\mu} dy}{V(1-yy)}$$

substituamus jam pro μ successive valores 0, 2, 4, 6, 8 &c. & quoniam vidimus esse $\int \frac{dy}{V(1-yy)} = \frac{\pi}{2}$, erit, ut sequitur:

$$\text{si } \mu = 0; \int \frac{y^0 dy}{V(1-yy)} = \frac{1}{2} \int \frac{dy}{V(1-yy)} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$(\mu = 2; \int \frac{y^2 dy}{V(1-yy)} = \frac{3}{4} \int \frac{y^0 dy}{V(1-yy)} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2}$$

$$\mu = 4; \int \frac{y^4 dy}{V(1-yy)} = \frac{5}{6} \int \frac{y^2 dy}{V(1-yy)} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2}$$

$$(\mu = 6; \int \frac{y^6 dy}{V(1-yy)} = \frac{7}{8} \int \frac{y^4 dy}{V(1-yy)} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}$$

unde lex, qua sequentes progrediuntur, sponte elucet.

§. XV. Quodsi jam isti valores pro formulis integralibus, ex quibus longitudo quadrantis elliptici q consari inventus est, substituantur, reperitur

$$q = \frac{\pi}{2} - \frac{1}{2} \pi n. \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1 \cdot 1}{2 \cdot 4} \pi^2. \frac{1 \cdot 3}{2 \cdot 4} \frac{\pi}{2} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \pi^3. \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{\pi}{2}$$

quæ ad sequentem seriem satis concinnam revocatur

$$q = \frac{\pi}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} n^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} n^4 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 - \&c. \right)$$

cujus



cujus lex progressionis est manifesta. Restituatur ergo pro nn suus valor $1 - pp$, eritque

$$q = \frac{\pi}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} (1 - pp) - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} (1 - pp)^3 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} (1 - pp)^5 - \&c. \right)$$

§. XVI. Cum pro curva nostra AQDq littera p exhibeat abscissam CP & littera q applicatam PQ; jam adepti sumus pro ista curva æquationem inter ejus coordinatas p & q , quæ etsi serie constat infinita, tamen non solum ejus naturam in se complectitur, sed etiam valores applicatæ q mox satis accurate exhibet, si abscissa p parum ab unitate differat: hoc est, cum sit $CB = CA = 1$, si punctum P ipsi B fuerit proximum; tum enim ob $1 - pp = nn$ quantitatem valde parvam series inventa valde convergit.

§. XVII. Hinc igitur indolem nostræ curvæ prope punctum D, hoc est ejus directionem & curvaturam definire poterimus.

Primo enim patet uti jam vidimus, si $p = 1$ fore $q = \frac{\pi}{2}$, ita ut

sumpta abscissa $CB = 1$ sit applicata $BD = \frac{\pi}{2} = 1,570796326$

7948961. Deinde ad positionem tangentis inveniendam, quæ-ratur ratio differentialium $dq:dp$, quæ per differentiationem reperitur:

$$\frac{dq}{dp} = \frac{\pi}{2} p \left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} (1 - pp) + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} (1 - pp)^3 + \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 8} (1 - pp)^5 + \&c. \right)$$

Euleri Opuscula Tom. II.

R

Po-

Posito jam $p = 1$ fiet $\frac{dq}{dp} = \frac{\pi}{4}$. Unde si DG sit tangens curvæ in puncto D, cum sit BD: BG = dq: dp, erit BG = $\frac{dp}{dq}$. BD = $\frac{4}{\pi}$. BD & ob BD = $\frac{\pi}{2}$, fiet BH = 2 = 2BC, & CG = BC. Sicque hoc casu subtangens BG erit dupla abscissæ BC: & cum anguli BGD tangens sit = $\frac{dq}{dp} = \frac{\pi}{4} = 0,78539816$ erit angulus BGD = $\overset{\circ}{38}, \overset{'}{8}, \overset{''}{45}, \overset{'''}{41}, \overset{IV}{51}$.

§. XVIII. Ad radium osculi seu evolutæ in puncto D definiendum, cum sit ob $\frac{dq}{dp} = \frac{\pi}{4}$, elementum curvæ $\sqrt{(dp^2 + dq^2)} = dp \sqrt{(1 + \frac{\pi^2}{16})}$, erit radius osculi = $(1 + \frac{\pi^2}{16})^{\frac{3}{2}} dp : d dq$.

At sumendis differentialibus secundis erit

$$\frac{ddq}{dp^2} = \frac{\pi}{2} \left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} (1 - pp) + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 6} (1 - pp)^2 + \&c \right) - \frac{\pi}{2} pp \left(\frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 6} (1 - pp) + \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 7} (1 - pp)^2 + \&c \right)$$

Posito ergo $p = 1$, erit $\frac{ddq}{dp^2} = \frac{\pi}{2} \left(\frac{1}{2} - \frac{3}{8} \right) = \frac{\pi}{16}$. Unde in puncto curvæ D erit radius evolutæ = $\frac{16}{\pi} \left(1 + \frac{\pi^2}{16} \right) \sqrt{(1 + \frac{\pi^2}{16})}$, qui valor in numeris proxime reperitur = 10,470672.

§. XIX.

§. XIX. Potest hinc adhuc alia series inveniri, quæ valorē applicatæ $PQ = q$ exprimat. Consideretur, enim illud altorum curvæ punctum q , pro quo sit abscissa $Cp = P$ & applicata $pq = Q$, erit quoque ob $P > 1$.

$$Q = \frac{\pi}{2} \left(1 + \frac{1.1}{2.2} (PP-1) - \frac{1.1.1.3}{2.2.4.4} (PP-1)^2 + \right. \\ \left. \frac{1.1.1.3.5}{2.2.4.4.6.6} (PP-1)^3 - \&c. \right).$$

Jam vero supra notavimus, si sit $P = \frac{1}{p}$, fore $Q = \frac{q}{p}$; quare his valoribus substitutis impetrabimus novam æquationem inter p & q , qua natura curvæ pariter exprimetur.

$$q = \frac{\pi}{2} p \left(1 + \frac{1.1}{2.2} \frac{(1-pp)}{pp} - \frac{1.1.1.3}{2.2.4.4} \frac{(1-pp)^2}{p^4} + \right. \\ \left. \frac{1.1.1.3.5}{2.2.4.4.6.6} \frac{(1-pp)^3}{p^6} - \&c. \right)$$

quæ si cum ante inventa combinetur, innumerabiles aliæ novæ æquationes obtineri poterunt. Veluti si prior per p multiplicata ab hac subtrahatur, prodibit.

$$q - pq = \frac{\pi}{2} p \left(\frac{1.1}{2.2} \frac{(1-pp)(1+pp)}{pp} - \frac{1.1.1.3}{2.2.4.4} \right. \\ \left. \frac{(1-pp)(1-p^4)}{p^4} + \&c. \right)$$

quæ reducitur ad hanc:

R 2

q =

§. XIX.

$$q = \frac{\pi}{4} (1+p) \left(\frac{1}{2} \cdot \frac{1+pp}{p} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} \frac{(1-p^4)(1-pp)}{p^3} + \right. \\ \left. \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \frac{(1+p^6)(1-pp)^2}{p^5} - \&c. \right).$$

vel cum series adhuc sit divisibilis per $\frac{1+pp}{2p}$ erit

$$q = \frac{\pi}{8} \cdot \frac{(1+p)(1+pp)}{p} \left(1 - \frac{1 \cdot 3}{4 \cdot 4} \frac{(1-pp)}{pp} (1-pp) + \frac{1 \cdot 3 \cdot 5}{4 \cdot 4 \cdot 6 \cdot 6} \right. \\ \left. \frac{(1-pp+p^4)}{p^4} (1-pp)^2 - \frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 4 \cdot 6 \cdot 8 \cdot 8} \frac{(1-pp+p^4-p^6)}{p^6} (1-pp)^3 \right. \\ \left. + \&c. \right)$$

§. XX. Manifestum autem est has series parum subsidii afferre, si applicatas invenire velimus, quæ longius a BD, quæ abscissæ $p = 1$ respondet, sint remotæ, si enim pro p ponatur numerus vel valde magnus vel valde parvus, series inventa vel parum admodum convergit vel etiam divergit. Si enim inde longitudinem primæ applicatæ CA, quæ abscissæ $p = 0$ respondet, definire velimus, serie primum inventa uti conveniet, quia in reliquis termini evadunt infinite magni. Habebimus igitur pro hoc casu $p = 0$;

$$q = \frac{\pi}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \right. \\ \left. \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 8 \cdot 8} - \&c. \right)$$

quæ tam lente convergit, ut etiam si plurimi termini actu colligerentur, tamen verus ipsius q valor, quem novimus esse $= 1$, inde difficillime agnosci posset.

§. XXI.



§. XXI. Quanquam autem nunc quidem novimus esse

$$1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \&c. = \frac{2}{\pi}$$

tamen inventio summæ hujus seriei non parum ardua videtur, si a priori tentetur. Veritatem quidem ex formula, quam quondam Wallisius pro circuli quadratura dedit, intelligere licet, si termini ab initio in unum colligantur, sic enim prodit

$$1 - \frac{1 \cdot 1}{2 \cdot 2} = \frac{1 \cdot 3}{2 \cdot 2}$$

$$\frac{1 \cdot 3}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} = \frac{1 \cdot 3 \cdot (4 \cdot 4 - 1 \cdot 1)}{2 \cdot 2 \cdot 4 \cdot 4} = \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4}$$

$$\frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}$$

unde valor seriei in infinitum continuatæ erit

$$\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot 13}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot 14} - \&c.$$

quæ expressio cum sit ipsa Wallisiana, patet summam nostræ seriei esse $= \frac{2}{\pi}$. Interim tamen juvabit tradere methodum hanc seriem aliasque similes a priori summandi.

Problema.

Invenire summam hujus seriei infinitæ:

$$1 - \frac{1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \&c.$$

ujus lex progressionis primo intuitu est manifesta.

Solutio.

§. XXII. Ponatur summa hujus serici, quæ quæritur = s , ut sit

$$s = 1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \&c.$$

Jam eligatur series, cujus summa constet, & cujus coefficientes jam in his terminis contineantur. Cujusmodi est hæc

$$\frac{1}{\sqrt{1-xx}} = 1 + \frac{1}{2}xx + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^8 + \&c.$$

Erit ergo per differentiale quodpiam dP multiplicando & integrando

$$\int \frac{dP}{\sqrt{1-xx}} = P + \frac{1}{2} \int xx dP + \frac{1 \cdot 3}{2 \cdot 4} \int x^4 dP + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \int x^6 dP + \&c.$$

Nunc differentiale hoc dP ita definiatur, ut si post integrationem ponatur $x = 1$. fiat

$$\int x dP = -\frac{1}{2}P$$

$$\int x^4 dP = +\frac{1}{4} \int x x dP = -\frac{1 \cdot 1}{2 \cdot 4}P$$

$$\int x^6 dP = +\frac{3}{8} \int x^4 dP = -\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}P$$

$$\int x^8 dP = +\frac{5}{16} \int x^6 dP = -\frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}P$$

quo facto si hi valores substituantur habebitur;

$$\int \frac{dP}{\sqrt{1-xx}} = P \left(1 - \frac{1 \cdot 1}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \&c. \right)$$

ideoque

itur = 1, ut

&c.

efficientes jam

$$\frac{1.3.5.7}{2.4.6.8} x +$$

integrando

$$x dP + \&c.$$

egrationem

ideoque $\frac{dP}{\sqrt{1-xx}} = P$, si quidem post integrationem statuatur $x = 1$.

§. XXIII. Huc ergo res redit, ut quaeratur formula differentialis dP , ut superioribus conditionibus satisfiat.

seu ut in genere sit $x^{\mu+2} dP = \frac{\mu-1}{\mu-2} x^{\mu} dP$, si quidem post integrationem utramque ponatur $x = 1$. Omissa igitur hac conditione sit

$$x^{\mu+2} dP = \frac{\mu-1}{\mu+2} x^{\mu} dP + \frac{Qx^{\mu+2}}{\mu+2}$$

ubi Q ejus modi sit functio ipsius x , quae evanescat posito $x = 1$.

Capiantur ergo differentialia, eritque per x^{μ} dividendo

$$xxdP = \frac{\mu-1}{\mu+2} dP + \frac{xdQ + (\mu+1)Qdx}{\mu+2}$$

seu $0 = (\mu-1)dP - (\mu+2)xxdP + xdQ + (\mu+1)Qdx$ quae aequatio, cum locum habere debeat pro omni valore ipsius μ , resolvetur in has duas:

$$0 = dP - xxdP + Qdx$$

$$0 = -dP - 2xxdP + xdQ + Qdx$$

$$\text{unde sit } dP = \frac{-Qdx}{1-xx} = \frac{xdQ + Qdx}{1+2xx}$$

$$\& \quad xdQ(1-xx) = -Qdx(2+xx)$$

— &c)

ideoque

Quae



Quare cum sit $\frac{dQ}{Q} = \frac{dx(1+xx)}{x(1-xx)} = -\frac{2dx}{x} - \frac{3x dx}{1-xx}$

erit $Q = \frac{-(1-xx)^{\frac{3}{2}}}{xx}$ & $dP = \frac{dx}{xx} \sqrt{1-xx}$

§. XXIV. Verum hic notandum est, etsi valor ipsius Q evanescat posito $x = 1$; tamen casu $\mu = 0$, quantitatem algebraicam $\mu + 1$

$\frac{Qx}{\mu + 2}$ non evanescere, si ponatur $x = 0$, quæ tamen conditio æque est necessaria atque altera, ita ut hoc casu non sit $\int xxdP = -\frac{1}{2}P$. Cum autem reliquæ formulæ quibus $\mu > 0$ locum habeant, a formula $\int xxdP$ erit incipiendum, eritque

$$\int x^4 dP = \frac{1}{4} \int xxdP$$

$$\int x^6 dP = \frac{3}{6} \int x^4 dP = \frac{1 \cdot 3}{4 \cdot 6} \int xxdP$$

$$\int x^8 dP = \frac{5}{8} \int x^6 dP = \frac{1 \cdot 3 \cdot 5}{4 \cdot 6 \cdot 8} \int xxdP$$

&c.

unde habebitur

$$\int \frac{dP}{\sqrt{1-xx}} = P + \int xxdP \left(\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. \right)$$

At est $\frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \& = 2(1-s)$; ideoque

$\int dP$



$$-\frac{3xdr}{1-xx}$$

$$1-xx)$$

ipſius Q evanescit
acem algebraicam

non condicio æque

et $\int xrdP = -\frac{1}{2}P$.
n habeant, a for-

$$\int \frac{dP}{V'(1-xx)} = P + 2(1-x) \int xxdP. \text{ At ob } dP = \frac{dx}{xx} V(1-xx)$$

$$\text{erit } P = C - \frac{V(1-xx)}{x} - A \sin x;$$

$$\int xxdP = \int dx V(1-xx) =$$

$$\frac{1}{2} A \sin x + \frac{1}{2} x V(1-xx), \text{ \& } \int \frac{dP}{V(1-xx)} = D - \frac{1}{x}$$

ubi constantes C & D ita accipi debent, ut integralia hæc evanescant posito $x = 0$: quanquam autem utraque seorsim sit infinita, tamen conjunctæ se mutuo destruent. Erit enim

$$\int \frac{dP}{V(1-xx)} - P = D - \frac{1}{x} - C + \frac{V(1-xx)}{x} + A \sin x$$

quæ ut evanescat posito $x = 0$, debet esse $D = C$, ideoque posito

$$\text{jam } x = 1 \text{ fiet } \int \frac{dP}{V(1-xx)} - P = -1 + \frac{\pi}{2}: \text{ \& quia eo-}$$

$$\text{dem hoc casu est } \int xxdP = \frac{\pi}{4}, \text{ prodibit}$$

$$-1 + \frac{\pi}{2} = 2(1-x) \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{2},$$

$$\text{hincque colligitur fore } \frac{\pi}{2} = 1 \text{ \& } x = \frac{2}{\pi} \text{ seu}$$

$$\frac{1.3.3.5}{4.6.6} + \&c.)$$

$$1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \& = \frac{2}{\pi}$$

); ideoque

$\int dP$

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S

§. XXX.



§. XXV. Quoniam igitur eruius in ipso initio esse applicatam curvæ CA = 1, indolem hujus curvæ prope punctum A indagemus, seu in valorem applicatæ q inquiremus, si abscissa p fuerit valde parva. In hunc finem ponamus iterum $1 - pp = nn$, & cum sit

$$q = \frac{\pi}{2} \left(1 - \frac{1.1}{2.2} nn - \frac{1.1.1.3}{2.2.4.4} n^4 - \frac{1.1.1.3.5}{2.2.4.4.6.6} n^6 - \&c. \right)$$

& quia novimus fore proxime $q = 1$, addamus æqualitatem modo inventam:

$$0 = 1 - \frac{\pi}{2} \left(1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.5}{2.2.4.4.6.6} n^6 - \&c. \right)$$

atque habebimus:

$$q = 1 + \frac{\pi}{2} \left(\frac{1.1}{2.2} (1 - nn) + \frac{1.1.1.3}{2.2.4.4} (n^4 - n^6) + \frac{1.1.1.3.5}{2.2.4.4.6.6} (1 - n^6) + \&c. \right)$$

cujus seriei cum singuli termini sint per $1 - nn = pp$ divisibiles, re-ducetur hæc expressio ad hanc:

$$q = 1 + \frac{\pi}{8} pp \left(1 + \frac{1.3}{4.4} (1 + nn) + \frac{1.3.3.5}{4.4.6.6} (1 + nn + n^4) + \frac{1.1.3.3.5.7}{4.4.6.6.8.8} (1 + n^2 + n^4 + n^6) + \&c. \right)$$

§. XXVI. Quodsi in hac expressione singuli termini ad potestates ipsius n evolvantur, reperietur

+



$$q = 1 + \frac{\pi}{2} fp \left\{ \begin{array}{l} + \frac{1 \cdot 1}{2 \cdot 2} + \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \\ \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8} + \&c. \\ + \frac{1}{n} \left(\frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 4}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \right. \\ \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8} + \&c. \\ + \frac{1}{n^2} \left(\frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8} + \right. \\ \&c. \\ + \frac{1}{n^3} \left(\frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8} + \&c. \right. \\ \&c. \end{array} \right.$$

At ex supra inventis habemus summam primæ seriei

$$\frac{1 \cdot 1}{2 \cdot 2} + \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. = 1 - \frac{2}{\pi}$$

quæ si primo termino multetur, prodibit secunda, quæ est coefficientens ipsius πn , ita ut sit

$$\frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. = \frac{1 \cdot 3}{2 \cdot 2} - \frac{2}{\pi}$$

hæc denuo primo termino multata, dabit coefficientem ipsius n^2 , nempe

$$\frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} + \&c. = \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{2}{\pi}$$

S 2

simi-



similique modo coefficientis ipsius n^6 erit $= \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \frac{2}{\pi}$

& ita porro, sicque tandem obtinebitur.

$$q = 1 + \frac{\pi}{2} pp \left(\left(1 - \frac{2}{\pi} \right) + \left(\frac{1 \cdot 3}{2 \cdot 2} - \frac{2}{\pi} \right) nn + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{2}{\pi} \right) n^4 + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \frac{2}{\pi} \right) n^6 + \&c. \right)$$

vel erit;

$$q = 1 + pp \left(\left(\frac{\pi}{2} - 1 \right) + \left(\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} - 1 \right) nn + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4} \cdot \frac{\pi}{2} - 1 \right) n^4 + \left(\frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \cdot \frac{\pi}{2} - 1 \right) n^6 + \&c. \right)$$

§. XXVII. Ponamus jam hic $n = 1$, ut obtineamus æquationem hujus formæ $q = 1 + App$, qua natura curvæ prope punctum A exprimitur: cum enim conjectare liceat veram æquationem futuram esse hujus formæ:

$$q = 1 + App + Bp^4 + Cp^6 + Dp^8 + \&c.$$

si abscissa p valde parva assumatur, reliqui termini præter binos primos omitti poterunt, atque ex æquatione $q = 1 + App$, tam positio tangentis, quam curvatura in puncto A colligi poterit. Posito enim $AR = x$, $RQ = y$, erit $q = 1 + x$ & $p = y$, ideoque si arcus AQ fuerit minimus, is cum parabola confundetur, cujus æquatio $x = Ayy$ seu $yy = \frac{1}{A}x$, ac propterea $\frac{1}{A}$ parameter. Unde sequitur tangentem curvæ in A fore ad rectam AC perpendicularem, & radium osculi ibidem esse $= \frac{1}{2A}$.

§. XXVIII.



§. XXVIII. Hic igitur coefficientis A reperietur, si in superiore serie, per quam quantitas pp multiplicatur, ponatur $n = 1$; ita ut sit

$$A = \left(\frac{\pi}{2} - 1\right) + \left(\frac{1.2}{2.2} \cdot \frac{\pi}{2} - 1\right) + \left(\frac{1.3.3.5}{2.2.4.4} \cdot \frac{\pi}{2} - 1\right) + \&c.$$

quæ autem si ejus summatio tentetur, tam parum convergens deprehenditur, ut ejus summam adeo infinitam suspicari debeamus. In hac autem suspitione eo magis confirmamur, si seriem primo (§. 15.) inventam, secundum dimensiones ipsius p evolamus, unde fit

$$= \frac{\pi}{2} \left\{ \begin{aligned} &1 - \frac{1.1}{2.2} - \frac{1.1.1.3}{2.2.4.4} - \frac{1.1.1.3.3.5}{2.2.4.4.6.6} - \&c. \\ &+ pp \left(\frac{1.1}{2.2} + \frac{1.1.1.2}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} \cdot 3 + \&c. \right) \\ &- p^2 \left(\frac{1.1.1.3}{2.2.4.4} + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} \cdot 3 + \&c. \right) \end{aligned} \right.$$

§. XXIX. Hinc ergo coefficientis ipsius pp in æquatione generali pro curva $q = 1 + Afp + Bp^2 + Cp^3 + Dp^4 + \&c.$ erit

$$A = \frac{\pi}{2} \left(\frac{1.1}{2.2} \cdot 1 + \frac{1.1.1.3}{2.2.4.4} \cdot 2 + \frac{1.1.1.3.3.5}{2.2.4.4.6.6} \cdot 3 + \&c. \right)$$

$$\text{seu } A = \frac{\pi}{4} \left(\frac{1}{2} + \frac{1.1.3}{2.2.4} + \frac{1.1.3.3.5}{2.2.4.4.6} + \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8} + \&c. \right)$$

similique modo & reliquos coefficientes B, C, D, &c. ex hac serie eruere licebit. Verum hoc labore superfedere poterimus, cum liqueat non solum coefficientem A, sed etiam omnes reliquos produ-



dituros esse infinitos. Perspicuum hoc fiet ex solutione hujus problematis.

Problema.

Invenire summam hujus seriei infinitæ:

$$S = \frac{1}{2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 6} + \&c.$$

Solutio.

§. XXX. Assumatur ad hanc summam s inveniendam hæc formula:

$$\frac{1}{\sqrt{1-xx}} = 1 + \frac{1}{2}xx + \frac{1 \cdot 2}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \&c.$$

ut fit

$$\int \frac{dP}{\sqrt{1-xx}} = P + \frac{1}{2} \int xx dP + \frac{1 \cdot 3}{2 \cdot 4} \int x^4 dP + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \int x^6 dP + \&c.$$

fitque si post integrationes singulas ponatur $x = 1$

$$\int xx dP = \frac{3}{4} P$$

$$\int x^4 dP = \frac{5}{6} \int xx dP = \frac{3 \cdot 5}{4 \cdot 6} P$$

$$\int x^6 dP = \frac{5}{8} \int x^4 dP = \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} P$$

&c.

hincque fiet

$\int dP$

ione hujus pro-

$$\frac{dP}{\sqrt{(1-xx)}} = P \left(1 + \frac{1 \cdot 3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \&c. \right)$$

sive $\int \frac{dP}{\sqrt{(1-xx)}} = 2Ps$; unde invento P , reperietur s post integrationem ponatur $x = 1$.

§. XXXI. Cum igitur generaliter esse debeat.

$$\int x^{\mu+1} dP = \frac{\mu+3}{\mu+4} \int x^{\mu} dP + \frac{x^{\mu+1} Q}{\mu+4}$$

dummodo Q ejusmodi sit functio, quæ evanescat posito $x = 1$, erit

$$(\mu+4) xxdP = (\mu+3) dP + x dQ + (\mu+1) Qdx$$

unde duæ sequentes æquationes consociuntur.

$$xxdP = dP + Qdx$$

$$4xxdP = 3dP + dQ + Qdx$$

$$\& dP = \frac{-Qdx}{1-xx} = \frac{-xdQ - Qdx}{3-4xx}$$

$$\text{hincque elicitur } \frac{dQ}{Q} = \frac{2dx - 3xxdx}{x(1-xx)} = \frac{2dx}{x} - \frac{xdx}{1-xx}$$

& $Q = -xx\sqrt{(1-xx)}$. Quare habebitur

$$dP = \frac{xxdx}{\sqrt{(1-xx)}} \& \frac{dP}{\sqrt{(1-xx)}} = \frac{xxdx}{1-xx} = -dx + \frac{dx}{1-xx}.$$

Fiet ergo $P = \frac{1}{2}\pi$, si post integrationem ponatur $x = 1$

ac

$\int dP$

at $\int \frac{dP}{\sqrt{1-xx}} = x + \frac{1}{2} \frac{1+x}{1-x}$, cujus valor posito $x = 1$ fit utique infinitus. Erit igitur $s = \infty$, seu summa seriei propositæ infinite magna.

§. XXXII. Quia igitur coefficientes A ipsius pp , in æquatione

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + \&c.$$

est infinitus, radius osculi curvæ in puncto A utique erit infinite parvus. Verum præterea hæc æquatio: in qua omnes omnino coefficientes A, B, C, D &c. sunt infiniti, nihil plane ad curvæ cognitionem confert. Quia enim radius osculi curvæ in A est infinite parvus, natura curvæ circa punctum A hujus modi æquati-

one $q = 1 + \alpha p^m$ exprimitur, in qua exponens m binario sit minor, verumtamen unitate major: sed ex omnibus, quæ hætenus sunt tradita nulla via patet, qua hunc exponentem m scrutari queamus. Cum enim is numerus integer esse nequeat, nulla serierum, quas pro q eruimus, ita est comparatæ, usque ex ea potestatem ipsius p irrationalem elicere liceat.

§. XXXIII. Hinc intelligimus problema esse summo opere difficile, quo æquatio tantum elementaris requiritur, quæ naturam curvæ propositæ AQD saltem proxime circa punctum A exhibeat. Notum est enim si ponatur $AR = x$ & $RQ = y$, quæcunque fuerit curva AQ , naturam minimæ ejus portiunculæ cir-

ca A semper hujusmodi æquatione $y = Ax$, comprehendendi posse; siquidem curva sit algebraica; pro curvis autem transcendentibus certum videtur, quasvis earum minimas portiunculas cum arcubus curvarum algebraicarum comparari posse. Quare in nostra curva, etsi

etfi est transcendens, hoc eo magis mirum videri debet, quod nul-

la hujusmodi formula $y = Ax^m$ exhiberi possit, quæ saltem minimæ ejus portiunculæ circa A sitæ naturam declaret.

§. XXXIV. Hunc nodum ut resolvamus, æquationem nobis finitam inter coordinatas p & q investigare oportebit, quæ etfi, ut facile prævidere licet, ad differentialia secundi ordinis exsurget, tamen ad accuratiorem curvæ cognitionem magis erit acomodata. Eliciemus autem hujusmodi æquationem, quæ numero terminorum finito constet, si seriem primo inventam ad summam revocabimus. Cum enim posito $1 - pp = mn$ sit

$$\frac{2q}{\pi} = 1 - \frac{1.1}{2.2} nn - \frac{1.1.1.3}{2.2.4.4} n^4 - \frac{1.1.1.3.5}{2.2.4.4.6.6} n^6 - \&c.$$

Erit differentiando

$$\frac{2dq}{\pi dn} = -\frac{1.1}{2} n - \frac{1.1.1.3}{2.2.4} n^3 - \frac{1.1.1.3.5}{2.2.4.6} n^5 - \&c.$$

quæ per n multiplicata denuoque differentiatâ dat

$$\frac{2}{\pi} \frac{d}{dn} \frac{ndq}{dn} = -1.1n - \frac{1.1}{2.2} 1.3n^3 - \frac{1.1.1.3}{2.2.4.4} 3.5n^5 - \&c.$$

Multiplicetur hæc per $\frac{dn}{n}$ ac rursus integretur, erit

$$\frac{2}{\pi} \int \frac{1}{n} d. \frac{ndq}{dn} = -1.n - \frac{1.1}{2.2} 1.n^3 - \frac{1.1.1.3}{2.2.4.4} 3.n^5 - \&c.$$

Multiplicetur per $\frac{dn}{n^3}$ & integrando prodibit,

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T

$\frac{2}{\pi}$

$$\frac{2}{\pi} \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{1}{n} - \frac{1.1}{2.2} n - \frac{1.1.1.3}{2.2.4.4} n^3 - \&c.$$

quæ series cum sit ipsa proposita, per n divisa erit

$$\frac{2}{\pi} \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{2q}{\pi n} \text{ seu } \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{q}{n}$$

§. XXXV. Sumamus nunc differentialia, habebiturque

$$\frac{dn}{n^2} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{ndq - qdn}{nn} \text{ seu}$$

$$\int \frac{1}{n} d. \frac{ndq}{dn} = \frac{nn dq}{dn} - nq.$$

porroque differentiando

$$\frac{1}{n} d. \frac{ndq}{dn} = nd. \frac{ndq}{dn} + ndq - ndq - qdn$$

$$\text{seu } (1 - nn) d. \frac{ndq}{dn} + qndn = 0$$

$$\text{Jam ob } 1 - nn = pp \text{ erit } ndn = -pdp \& \frac{dn}{n} = -\frac{pdp}{1 - pp}$$

$$\text{unde fit } -ppd. \frac{(1 - pp) dq}{pdp} - pqdp = 0 \text{ seu}$$

$$d. \frac{(1 - pp) dq}{pdp} + \frac{qdp}{p} = 0. \text{ Sumatur jam } dp \text{ constans erit}$$

$$\frac{(1 - pp) dq}{pdp} - \frac{dq (1 + pp)}{pp} + \frac{qdp}{p} = 0 \text{ seu}$$

$$p. (1 - pp) ddq - dpdq (1 + pp) + pqdp = 0.$$

§. XXXVI.



§. XXXVI. En igitur æquationem differentialem secundæ gradus pro curva proposita

$$p(1 - pp) dq - dpdq(1 + pp) + pqdp = 0$$

ex qua potestas illa ipsius p in æquatione $q = 1 + Ap^m$ elici debet, si abscissa p valde parva statuatur. Cum igitur fiat $dq = m$

$$Ap^{m-1} dp \text{ \& } ddq = m(m-1) Ap^{m-2} dp \text{ orietur}$$

$$\left. \begin{aligned} m(m-1) Ap^{m-1} - mAp^{m-1} + p \\ - m(m-1) Ap^{m+1} - mAp^{m+1} + Ap^{m+1} \end{aligned} \right\} = 0.$$

$$\text{scu } m(m-2) Ap^{m-1} - mmAp^{m+1} + p = 0.$$

Deberet ergo esse $m = 2$, ut terminus Ap^{m-1} cum p comparari posset, sed tum iterum obtinetur $A = \infty$: præterea vero hinc perspicitur exponentem m nullo modo numerum fractum esse posse, ita ut hinc difficultas supra memorata augeri potius quam tolli videatur.

§. XXXVII. Quodsi regulis consuetis uti velimus ad æquationem inventam in seriem evolvendam, quæ secundum potestates ipsius p procedat, quoniam novimus primum seriei terminum esse $= 1$, nullam aliam formam inde colligere licet nisi hanc:

$$q = 1 + Ap + Bp^2 + Cp^3 + Dp^4 + \&c.$$

unde fit

$$\frac{dq}{dp} = 2Ap + 4Bp^2 + 6Cp^3 + 8Dp^4 + \&c.$$



$$\& \frac{ddq}{dp^3} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$$

qui valores in æquatione substituti præbunt:

$$\left. \begin{aligned} &+ 2Ap + 12Bp^3 + 30Cp^5 + 56Dp^7 + \&c. \\ &- 2A - 12B - 30C - \&c. \\ &- 2Ap - 4B - 6C - 8D - \&c. \\ &- 2A - 4B - 6C - \&c. \\ &+ p + A + B + C + \&c. \end{aligned} \right\} = 0$$

unde omnes coefficientes A, B, C, &c. prodeunt infiniti.

§. XXXVIII. Hinc igitur videmus regulas ordinarias, secundum quas vulgo forma seriei, in quam æquatio differentialis transmutanda sit, dijudicari solet, non esse sufficientes, cum hoc casu nullam afferant utilitatem: unde nostra æquatio eo majorem meretur attentionem. Sequenti tamen modo ex ea natura curvæ prope punctum A colligi poterit, ex quo simul intelligetur, quemadmodum quoque in aliis casibus defectus iste regularum usu receptarum suppleri, eæque ad praxin accommodari debeant. Quia enim abscissam p hic pro infinite parva habemus, in æquatione pro $1 - pp$ & $1 + pp$ ponere licebit 1, & quia novimus esse hoc casu proxime $q = 1$, pro quantitate finita q unitatem scribamus: quo factò æquatio differentio-differentialis inventa pro casu, quo abscissa p est minima sequentem induet formam;

$$pddq - dpdq + pdp^2 = 0.$$

§. XXXIX. Hujus jam æquationis resolutio est facilis, cum enim dp sit constans, ponatur $dq = rdp$, erit $ddq = drdp$, habebiturque:

$$pdr - rdp + pdp = 0$$

pdr



$$\text{five } \frac{pdr - rdp}{pp} + \frac{dp}{p} = 0$$

cujus integrale est: $\frac{r}{p} + lp = C$, unde fit

$$r = Cp - plp$$

ideoque $dq = Cpdp - pdp lp$

Hæc jam æquatio integrata dabit:

$$q = 1 + \frac{1}{2} Cp^2 - pp lp + \frac{1}{2} pp^2$$

in qua cum terminus pp incomparabiliter sit minor quam pp^2 , erit pro curvæ initio A:

$$q = 1 - \frac{1}{2} pp lp$$

§. XL. Nuncigitur naturam curvæ prope initium A æquatione simplici definire possumus: si enim vocemus $AR = x$ & $RQ = y$, ob $p = y$ & $q = 1 + x$, orietur hæc $x = -\frac{1}{2} yy ly$, ad quam æquatio generalis pro curva revocatur, si coordinatæ x & y sint quam minimæ. Patet igitur ne minimum quidem arcum circa A tanquam portiunculam curvæ algebraicæ spectari posse, sed ejus naturam logarithmos implicare. Et quoniam æquatio logarithmica in exponentialem transformari potest, initium curvæ nostræ A commune erit cum linea transcendente, cujus æquatio est $e^{-2x} = yy$, sumto e pro numero cujus logarithmus hyperbolicus est 1.

§. XLI. Equatione hac $x = -\frac{1}{2} yy ly$ confirmantur quoque ea, quæ supra jam de affectionibus hujus curvæ in puncto A notavimus. Primo enim patet si fit $y = 0$, fore quoque $yy ly$ proinde $x = 0$, etsi hoc casu sit $ly = -\infty$. Deinde cum sit $dy = -y dy ly - \frac{1}{2} y dy$, quia y incomparabiliter est minus quam $yy ly$, erit

erit $dx = -y dy / ly$, ac propterea $\frac{dy}{dx} = \frac{-1}{y/ly} = \infty$ posito $y = 0$; unde patet tangentem curvæ in A ad abscissam AR esse perpendicularem. Porro cum sit subnormalis $\frac{y dy}{dx} = \frac{-1}{ly}$, hocque casu subnormalis radio evolutæ æquetur, ob $ly = \infty$ si $y = 0$, manifestum est radium osculi curvæ in A esse infinite parvum.

§. XLII. Maxime autem differt hæc curva a curvis algebraicis, quæ in initio A quoque habent radium osculi evanescentem. Curvarum enim algebraicarum, quæ hac indole gaudent, natura circa initium A hujusmodi formula exprimitur $x = ay^m$ existente $m < 2$, attamen $m > 1$. Sit igitur $m = 2 - s$ existente fractione unitate minore, ut sit $x = ay^{2-s}$, erit $dx = a(2-s)y^{1-s} dy$, ideoque $\frac{dy}{dx} = \frac{1}{a(1-s)y^{1-s}} = \infty$ ob $y^{1-s} = 0$: ac

radius osculi, qui subnormali $\frac{y dy}{dx}$ æqualis est, erit $= \frac{y}{a(2-s)}$ $= 0$. Pro nostra vero curva radius osculi inventus est $= \frac{-1}{ly}$, unde radius osculi evanescens in curva algebraica quacunq̃ue erit ad radium osculi in nostræ curvæ puncto A ut $-y / ly$ ad $a(2-s)$ hoc est ut 0 ad 1; quantumvis enim exiguus sit exponens s , casu $y = 0$ semper est $y / ly = 0$, etiamsi sit $ly = -\infty$. Quare in nostra quidem curva radius osculi in A est infinite parvus, sed tamen

men infinities major est, quam radius osculi evanescens in omni curva algebraica.

§. XLIII. Cognito jam initio seriei, qua valor applicatæ PQ = q per abscissam CP = p exprimitur, scilicet

$$q = 1 - \frac{1}{2} p p' p'' + A p p''$$

non difficile erit hinc formam totius seriei colligere. Cum enim ex æquatione differentio-differentiali intelligatur sequentium terminorum potestates ipsius p binario crescere, valor ipsius q generatim gemina serie infinita exprimitur, eritque

$$q = 1 + A p^2 + B p^4 + C p^6 + D p^8 + \&c.$$

$$- a p p' p'' - c p^4 p' - \frac{1}{2} p^6 p' + \delta p^8 p' - \&c.$$

in qua quidem nunc jam novimus esse $a = \frac{1}{2}$.

§. XLIV. Cum igitur verus valor ipsius q duplici serie contineatur, ut utramque seorsim eliciamus, ponamus

$$q = r - s p \quad \text{eritque differentiendo}$$

$$dq = dr - \frac{sdp}{p} - ds p$$

$$ddq = ddr - \frac{2dpds}{p} + \frac{sdp^2}{pp^2} - dds p$$

Hi valores in nostra æquatione differentiali

$$p(1 - pp) dq - dpdq(1 + pp) + p q dp = 0$$

substituuntur, ac termini per sp affecti seorsim nihilo æquantur, hoc modo duæ obtinebuntur æquationes:

I. q



$$\text{I. } p(1 - pp) dds \leftarrow (1 + pp) dpds + pids = 0$$

$$\text{II. } p(1 - pp) ddr - (1 + pp) dpdr + prdp - \\ 2(1 - pp) dpds + \frac{2idp}{p} = 0$$

§. XLV. Ad has æquationes resolvendas ponatur

$$r = 1 + Ap^3 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

$$s = ap^3 + \beta p^4 + \gamma p^5 + \delta p^6 + \epsilon p^8 + \&c.$$

eritque differentialibus sumendis

$$\frac{dr}{dp} = 2Ap^2 + 4Bp^3 + 6Cp^5 + 8Dp^7 + \&c.$$

$$\frac{ddr}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$$

$$\frac{ds}{dp} = 2ap^2 + 4\beta p^3 + 6\gamma p^4 + 8\delta p^5 + \&c.$$

$$\frac{dds}{dp^2} = 2a + 12\beta p^2 + 30\gamma p^4 + 56\delta p^6 + \&c.$$

His valoribus substitutis prima æquatio abibit in hanc:

$$\left. \begin{aligned} &2ap^3 + 12\beta p^4 + 30\gamma p^5 + 56\delta p^6 + 90\epsilon p^8 + \&c. \\ &\quad - 2a - 12\beta - 30\gamma - 56\delta - \&c. \\ -2a - 4\beta - 6\gamma - 8\delta - 10\epsilon - \&c. \\ &\quad - 2a - 4\beta - 6\gamma - 8\delta - \&c. \\ &\quad + a + \beta + \gamma + \delta + \&c. \end{aligned} \right\} = 0$$

§. XLVI.

§. XLVI. Si jam singularum potestatum ipsius p coefficientes nihilo æquales ponantur, erit:

$$2\alpha - 2\alpha = 0; \quad \alpha \text{ manet indeterminatum}$$

$$8\beta - 3\alpha = 0; \quad \beta = \frac{1 \cdot 3}{2 \cdot 4} \alpha$$

$$24\gamma - 15\beta = 0; \quad \gamma = \frac{3 \cdot 5}{4 \cdot 6} \beta = \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6} \alpha$$

$$48\delta - 35\gamma = 0; \quad \delta = \frac{5 \cdot 7}{6 \cdot 8} \gamma = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} \alpha$$

$$80\epsilon - 63\delta = 0; \quad \epsilon = \frac{7 \cdot 9}{8 \cdot 10} \delta = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10} \alpha$$

&c.

&c.

Si igitur valor coefficientis primi α constaret, quem quidem jam vidimus esse $= 1$, omnes sequentes coefficientes $\beta, \gamma, \delta, \epsilon$ forent cogniti. Verum resolutio alterius æquationis quoque hunc nobis valorem ipsius α patefaciet.

§. XLVII. Substitutis enim seriebus ante traditis in altera æquatione proveniet;

$ \begin{aligned} &2Ap + 12Bp^3 + 30Cp^5 + 56Dp^7 + 90Ep^9 + \&c. \\ &\quad - 2A - 12B - 30C - 56D - \&c. \\ &- 2A - 4B - 6C - 8D - 10E - \&c. \\ &\quad - 2A - 4B - 6C - 8D - \&c. \\ &+ 1 + A + B + C + D + \&c. \\ &- 4\alpha - 8\beta - 12\gamma - 16\delta - 20\epsilon - \&c. \\ &\quad + 4\alpha + 8\beta + 12\gamma + 16\delta + \&c. \\ &+ 2\alpha + 2\beta + 2\gamma + 2\delta + 2\epsilon + \&c. \end{aligned} $	$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} = 0$	$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} = 0$
<p>U</p>		<p>Unde</p>

Euleri Opuscula Tom. II.



Unde simili modo elicitur:

$$2A - 2A + 1 - 2\alpha = 0; \text{ hinc fit } \alpha = \frac{1}{2}$$

$$8B - 3A - 6\beta + 4\alpha = 0; 2.4 B - 1.3A + 2(2 - \frac{1.3.3}{2.4})\alpha = 0$$

$$24C - 15B - 10\gamma + 8\beta = 0; 4.6 C - 3.5B + 2(4 - \frac{3.5.5}{4.6})\beta = 0$$

$$48D - 35C - 14\delta + 12\gamma = 0; 6.8 D - 5.7C + 2(6 - \frac{5.7.7}{6.8})\gamma = 0$$

$$80E - 63D - 18\epsilon + 16\delta = 0; 8.10E - 7.9D + 2(8 - \frac{7.9.9}{8.10})\delta = 0$$

§. XLVIII. Cognito igitur valore ipsius $\alpha = \frac{1}{2}$ altera series s , quæ logarithmum ipsius p involvit, tota innotescit, erit enim:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{1.1.3}{2.2.4}$$

$$\gamma = \frac{1.1.3.3.5}{2.2.4.4.6}$$

$$\delta = \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8}$$

$$\epsilon = \frac{1.1.3.3.5.5.7.9}{2.2.4.4.6.6.8.10}$$

&c.

$$\text{fitque hinc } s = \alpha p p + \beta p^4 + \gamma p^6 + \delta p^8 + \epsilon p^{10} + \&c.$$

§. XLIX. Quod autem ad alteram seriem attinet

$$r = 1 + Ap^3 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \&c.$$

primus

primus coefficientis A hinc manet indeterminatus, cujus rei ratio est, quod has series ex æquatione differentiali secundi gradus elicuimus, quæ duplici determinatione indiget, ut ad nostrum casum accommodetur. Quare valorem hujus coefficientis A ex ipsa curvæ natura definiri oportet, eo autem invento, reliqui innotescunt ex his formulis, ad quas superiores redeunt.

$$B = \frac{1.3}{2.4} A - \frac{1}{2} \alpha \left(\frac{3}{2.2} + \frac{1}{1.1} \right)$$

$$C = \frac{3.5}{2.4} B - \frac{1}{2} \beta \left(\frac{3}{3.3} + \frac{1}{2.2} \right)$$

$$D = \frac{5.7}{6.8} C - \frac{1}{2} \gamma \left(\frac{3}{4.4} + \frac{1}{4.4} \right)$$

$$E = \frac{7.9}{8.10} D - \frac{1}{2} \delta \left(\frac{3}{5.5} + \frac{1}{4.4} \right)$$

§. L. His autem omnibus coefficientibus inventis ad datam quamvis abscissam GP = p, valor respondentis applicatæ PQ = q ita definitur, ut sit

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

$$- app'p - \beta p^4 p' - \gamma p^6 p' - \delta p^8 p' - \&c.$$

quæ series si abscissa p fuerit unitate multo minor, satis promte convergit, ut inde valor ipsius q cognosci queat. Hinc vero etiam applicatæ, quæ abscissis multo majoribus unitate respondent, definiri poterunt, quia abscissæ $\frac{1}{p}$ respondet applicata $\frac{q}{p}$. Quare si abscissa unitate multo major ponatur = P eique respondens applicata = Q ob $p = \frac{1}{P}$ & $q = p Q = \frac{Q}{P}$ erit

U 2

Q =

primus

$$Q = P + AP^{-1} + BP^{-2} + CP^{-3} + DP^{-4} + \&c. \\ + \alpha P^{-1} + \beta P^{-2} + \gamma P^{-3} + \delta P^{-4} + \epsilon P^{-5} + \&c.$$

Hinc si abscissa P fiat infinita erit

$$Q = P + \frac{\alpha P}{P} \text{ seu } Q - P = \frac{\alpha P}{P}$$

unde natura rami Dq in infinitum extensi & ad asymptotam CV appropinquantis colligitur.

§. LI. Quia porro novimus, si $p = 1$ fore $q = \frac{\pi}{2}$ pro hoc casu æquatio inventa hanc formam ob $1 =$ induet
 $\frac{\pi}{2} = 1 + A + B + C + D + E + \&c.$

Cum igitur valor A nondum sit definitus, reliqui vero $B, C, D \&c.$ ab eo pendeant, hæc æquatio conditionem continet, qua valor ipsius A determinatur. Ita scilicet valorem ipsius A comparatum esse oportet, ut summa seriei infinitæ $1 + A + B + C + \&c.$

fiat $\frac{\pi}{2}$. Verum si valores reliquarum litterarum $B, C, D \&c.$ qui ab A pendent, evolvantur, tam complicatæ resultant expressiones, ut hinc valor ipsius A nequitam erui possit.

§. LII. Ad hanc constantem A determinandam alia patet via, si datæ cujuscumque ellipsis perimeter ex altera formula in numeris fuerit inventa. Quæ methodus cum requirat, ut omnes coefficients in fractionibus decimalibus evolvantur, computo peracto reperietur;

“=

$a = 0,5000000000$ A queritur
 $\beta = 0,1673000000$; $B = 0,3750000000$ A — 0,1093750000
 $\gamma = 0,1171875000$; $C = 0,2343750000$ A — 0,0820312500
 $\delta = 0,0854492188$; $D = 0,1708984375$ A — 0,0641886393
 $\epsilon = 0,0672912598$; $E = 0,1345825195$ A — 0,0524978638
 $\zeta = 0,0555152893$; $F = 0,1110305786$ A — 0,0443481445
 $\eta = 0,0472540835$; $G = 0,0945081711$ A — 0,0383663416
 $\theta = 0,0411363691$; $H = 0,0822727382$ A — 0,037966962
 $i = 0,0364228268$; $I = 0,0728456536$ A — 0,0301949487
 $\kappa = 0,0326793696$; $K = 0,0653587392$ A — 0,0272843726
 &c.

Hisque valoribus inventis, si abscissa sit $CP = p$, valor applicatæ q ita definitur ut sit:

$$\begin{aligned}
 q = & 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + Gp^{14} + Hp^{16} + \\
 & Ip^{18} + Kp^{20} + \&c. \\
 & - p p I p. (a + \beta p^2 + \gamma p^4 + \delta p^6 + \epsilon p^8 + \zeta p^{10} + \eta p^{12} + \theta p^{14} + i p^{16} \\
 & + \kappa p^{18} + \&c.)
 \end{aligned}$$

§. LIII. Deinde vero supra ejusdem applicatæ q valorem ita invenimus expressum ut sit:

$$\begin{aligned}
 q = & \frac{\pi}{2} \left(1 - \frac{1.1}{2.2} (1 - pp) - \frac{1.1.1.3}{2.2.4.4} (1 - pp)^2 - \frac{1.1.1.3.5}{2.2.4.4.6.6} \right. \\
 & \left. (1 - pp)^3 + \&c. \right)
 \end{aligned}$$

Nunc igitur ex utraque formula pro eodem quopiam valore
U 3 re

$$\frac{2}{\pi} \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{1}{n} - \frac{1.1}{2.2} n - \frac{1.1.1.3}{2.2.4.4} n^3 - \&c.$$

quæ series cum sit ipsa proposita, per n divisa erit

$$\frac{2}{\pi} \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{2q}{\pi n} \text{ seu } \int \frac{dn}{n} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{q}{n}$$

§. XXXV. Sumamus nunc differentialia, habebiturque

$$\frac{dn}{n^2} \int \frac{1}{n} d. \frac{ndq}{dn} = \frac{ndq - qdn}{nn} \text{ seu}$$

$$\int \frac{1}{n} d. \frac{ndq}{dn} = \frac{nn dq}{dn} - nq.$$

porroque differentiando

$$\frac{1}{n} d. \frac{ndq}{dn} = nd. \frac{ndq}{dn} + ndq - ndq - qdn$$

$$\text{seu } (1 - nn) d. \frac{ndq}{dn} + qndn = 0$$

$$\text{Jam ob } 1 - nn = pp \text{ erit } ndn = -pdp \& \frac{dn}{n} = -\frac{pdp}{1 - pp}$$

$$\text{unde fit } -ppd. \frac{(1 - pp) dq}{pdp} - pqdp = 0 \text{ seu}$$

$$d. \frac{(1 - pp) dq}{pdp} + \frac{qdp}{p} = 0. \text{ Sumatur jam } dp \text{ constans erit}$$

$$\frac{(1 - pp) dq}{pdp} - \frac{dq (1 + pp)}{pp} + \frac{qdp}{p} = 0 \text{ seu}$$

$$p. (1 - pp) ddq - dpdq (1 + pp) + pqdp = 0.$$

§. XXXVI.



§. XXXVI. En igitur æquationem differentialem secundæ gradus pro curva proposita

$$p(1 - pp)ddq - dpdq(1 + pp) + pqdp^2 = 0$$

ex qua potestas illa ipsius p in æquatione $q = 1 + Ap^m$ elici debet, si abscissa p valde parva statuatur. Cum igitur fiat $dq = m$

$$Ap^{m-1} dp \& ddq = m(m-1)Ap^{m-2} dp^2 \text{ orietur}$$

$$\left. \begin{aligned} m(m-1)Ap^{m-1} - mAp^{m-1} + p \\ - m(m-1)Ap^{m+1} - mAp^{m+1} + p \end{aligned} \right\} = 0.$$

$$\text{scu } m(m-2)Ap^{m-1} - mmAp^{m+1} + p = 0.$$

Deberet ergo esse $m = 2$, ut terminus Ap^{m-1} cum p comparari posset, sed tum iterum obtinetur $A = \infty$: præterea vero hinc perspicitur exponentem m nullo modo numerum fractum esse posse, ita ut hinc difficultas supra memorata augeri potius quam tolli videatur.

§. XXXVII. Quod si regulis consuetis uti velimus ad æquationem inventam in seriem evolvendam, quæ secundum potestates ipsius p procedat, quoniam stovimus primum seriei terminum esse $\frac{1}{2}$, nullam aliam formam inde colligere licet nisi hanc:

$$q = 1 + Ap^{\frac{1}{2}} + Bp^{\frac{3}{2}} + Cp^{\frac{5}{2}} + Dp^{\frac{7}{2}} + \&c.$$

unde fit

$$\frac{dq}{dp} = 2Ap^{\frac{1}{2}} + 4Bp^{\frac{3}{2}} + 6Cp^{\frac{5}{2}} + 8Dp^{\frac{7}{2}} + \&c.$$



$$\& \frac{ddq}{dp^3} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \&c.$$

qui valores in æquatione substituti præbebunt :

$$\left. \begin{aligned} &+ 2Ap + 12Bp^3 + 30Cp^5 + 56Dp^7 + \&c. \\ &- 2A - 12B - 30C - \&c. \\ &- 2Ap - 4B - 6C - 8D - \&c. \\ &- 2A - 4B - 6C - \&c. \\ &+ p + A + B + C + \&c. \end{aligned} \right\} = 0$$

unde omnes coefficientes A, B, C, &c. prodeunt infiniti.

§. XXXVIII. Hinc igitur videmus regulas ordinarias, secundum quas vulgo forma seriei, in quam æquatio differentialis transmutanda sit, dijudicari solet, non esse sufficientes, cum hoc casu nullam afferant utilitatem: unde nostra æquatio eo majorem meretur attentionem. Sequenti tamen modo ex ea natura curvæ prope punctum A colligi poterit, ex quo simul intelligetur, quemadmodum quoque in aliis casibus defectus iste regularum usu receptarum suppleri, æque ad praxin accommodari debeant. Quia enim abscissam p hic pro infinite parva habemus, in æquatione pro $1 - pp$ & $1 + pp$ ponere licebit 1, & quia novimus esse hoc casu proxime $q = 1$, pro quantitate finita q unitatem scribamus: quo facto æquatio differentio-differentialis inventa pro casu, quo abscissa p est minima sequentem inducet formam;

$$pddq - dpdq + pdp^2 = 0.$$

§. XXXIX. Hujus jam æquationis resolutio est facilis, cum enim dp sit constans, ponatur $dq = rdp$, erit $ddq = drdp$, habebiturque:

$$pdr - rdp + pdp = 0$$

pdr



+ &c.

$$\text{five } \frac{pdr - rdp}{p^2} + \frac{dp}{p} = 0$$

cujus integrale est: $\frac{r}{p} + lp = C$, unde fit

$$r = Cp - p^2$$

$$\text{ideoque } dq = Cdp - pdp$$

Hæc jam æquatio integrata dabit:

$$q = 1 + \frac{1}{2} Cp^2 - pp + \frac{1}{2} p^2$$

in qua cum terminus pp incomparabiliter sit minor quam pp^2 , erit pro curvæ initio A:

$$q = 1 - \frac{1}{2} pp$$

unt infiniti.

regulas ordinarias
uatio differentialis
cientis, cum hoc
uatio eo majorem
do ex ea natura
simul intelligitur,
iste regularum ulu
modari debeant.
bemus, in æqua-
qua novimus esse
q unitatem scribe-
lis inventa pro a-
formam;

§. XL. Nuncigitur naturam curvæ prope initium A æquatione simplici definire possumus: si enim vocemus $AR = x$ & $RQ = y$, ob $p = y$ & $q = 1 + x$, orietur hæc $x = -\frac{1}{2} yy^2$, ad quam æquatio generalis pro curva revocatur, si coordinatæ x & y sint quam minimæ. Patet igitur ne minimum quidem arcum circa A tanquam portiunculam curvæ algebraicæ spectari posse, sed ejus naturam logarithmos implicare. Et quoniam æquatio logarithmica in exponentialem transformari potest, initium curvæ nostræ A commune erit cum linea transcendente, cujus æquatio

$$e^{2x} = y$$

est $e = y$, sumto e pro numero cujus logarithmus hyperbolicus est $= 1$.

io est facilis, cum
drdp, habebiturque:

§. XLI. Equatione hac $x = -\frac{1}{2} yy^2$ confirmantur quoque ea, quæ supra jam de affectionibus hujus curvæ in puncto A notavimus. Primo enim patet si sit $y = 0$, fore quoque yy^2 ac proinde $x = 0$, et si hoc casu sit $ly = 0$. Deinde cum sit $dy = -ydy$, quia y incomparabiliter est minus quam yy^2 ,



erit $dx = -y dy / ly$, ac propterea $\frac{dy}{dx} = -\frac{1}{y/ly} = \infty$ posito $y = 0$; unde patet tangentem curvæ in A ad abscissam AR esse perpendicularem. Porro cum sit subnormalis $\frac{y dy}{dx} = -\frac{1}{ly}$, hocque casu subnormalis radio evolutæ æquetur, ob $ly = \infty$ si $y = 0$, manifestum est radium osculi curvæ in A esse infinite parvum.

§. XLII. Maxime autem differt hæc curva a curvis algebraicis, quæ in initio A quoque habent radium osculi evanescentem. Curvarum enim algebraicarum, quæ hac indole gaudent,

natura circa initium A hujusmodi formula exprimitur $x = \alpha y^m$ existente $m < 2$, at tamen $m > 1$. Sit igitur $m = 2 - \alpha$ existente fractione unitate minore, ut sit $x = \alpha y^{2-\alpha}$, erit $dx = \alpha(2-\alpha)$

$y^{1-\alpha} dy$, ideoque $\frac{dy}{dx} = \frac{1}{\alpha(2-\alpha)y^{1-\alpha}} = \infty$ ob $y^{1-\alpha} = 0$: at

radius osculi, qui subnormali $\frac{y dy}{dx}$ æqualis est, erit $= \frac{y}{\alpha(2-\alpha)}$

$= 0$. Pro nostra vero curva radius osculi inventus est $= -\frac{1}{ly}$,

unde radius osculi evanescens in curva algebraica quacunque erit

ad radium osculi in nostræ curvæ puncto A ut $-y / ly$ ad $\alpha(2-\alpha)$ hoc est ut 0 ad 1; quantumvis enim exiguus sit exponens α , casu

$y = 0$ semper est $y / ly = 0$, etiam si sit $ly = -\infty$. Quare in nostra quidem curva radius osculi in A est infinite parvus, sed tamen

men infinities major est, quam radius osculi evanescens in omni curva algebraica.

§. XLIII. Cognito jam initio seriei, qua valor applicatæ $PQ = q$ per abscissam $CP = p$ exprimitur, scilicet

$$q = 1 - \frac{1}{2} p^2 + \frac{1}{24} p^4 - \frac{1}{720} p^6 + \dots$$

non difficile erit hinc formam totius seriei colligere. Cum enim ex æquatione differentio-differentiali intelligatur sequentium terminorum potestates ipsius p binario crescere, valor ipsius q generatim gemina serie infinita exprimitur, eritque

$$q = 1 - Ap^2 + Bp^4 - Cp^6 + Dp^8 - \dots$$

$$- \frac{1}{2} p^2 + \frac{1}{24} p^4 - \frac{1}{720} p^6 + \frac{1}{4200} p^8 - \dots$$

in qua quidem nunc jam novimus esse $\alpha = \frac{1}{2}$.

§. XLIV. Cum igitur verus valor ipsius q duplici serie contineatur, ut utramque seorsim eliciamus, ponamus

$$q = r - sp \quad \text{eritque differentiendo}$$

$$dq = dr - s dp - p ds$$

$$ddq = ddr - \frac{2dp ds}{p} + \frac{sdp^2}{pp} - dds p$$

Hi valores in nostra æquatione differentiali

$$p(1 - pp) ddq - dp dq (1 + pp) + p q dp^2 = 0$$

substituantur, ac termini per sp affecti seorsim nihilo æquantur, hoc modo duæ obtinebuntur æquationes:

I. q



$$\text{I. } p(1 - pp) \, ddr - (1 + pp) \, dpdr + p \, dp^2 = 0$$

$$\text{II. } p(1 - pp) \, ddr - (1 + pp) \, dpdr + p \, dp^2 -$$

$$2(1 - pp) \, dpdr + \frac{2sdp}{p} = 0$$

§. XLV. Ad has æquationes resolvendas ponatur

$$r = 1 + Ap^3 + Bp^4 + Cp^5 + Dp^6 + \&c.$$

$$s = ap^3 + \beta p^4 + \gamma p^5 + \delta p^6 + \epsilon p^7 + \&c.$$

eritque differentialibus sumendis

$$\frac{dr}{dp} = 2Ap + 4Bp^2 + 6Cp^3 + 8Dp^4 + \&c.$$

$$\frac{ddr}{dp^2} = 2A + 12Bp + 30Cp^2 + 56Dp^3 + \&c.$$

$$\frac{ds}{dp} = 2ap + 4\beta p^2 + 6\gamma p^3 + 8\delta p^4 + \&c.$$

$$\frac{dds}{dp^2} = 2a + 12\beta p + 30\gamma p^2 + 56\delta p^3 + \&c.$$

His valoribus substitutis prima æquatio abibit in hanc:

$$\left. \begin{aligned} &2ap + 12\beta p^2 + 30\gamma p^3 + 56\delta p^4 + 90\epsilon p^5 + \&c. \\ &- 2a - 12\beta - 30\gamma - 56\delta - \&c. \\ &- 2a - 4\beta - 6\gamma - 8\delta - 10\epsilon - \&c. \\ &- 2a - 4\beta - 6\gamma - 8\delta - \&c. \\ &+ a + \beta + \gamma + \delta + \&c. \end{aligned} \right\} = 0$$

§. XLVI.

§. XLVI. Si jam singularum potestatum ipsius p coefficientes nihilo æquales ponantur, erit:

$$2\alpha - 2\alpha = 0; \quad \alpha \text{ manet indeterminatum}$$

$$8\beta - 3\alpha = 0; \quad \beta = \frac{1 \cdot 3}{2 \cdot 4} \alpha$$

$$24\gamma - 15\beta = 0; \quad \gamma = \frac{3 \cdot 5}{4 \cdot 6} \beta = \frac{1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 4 \cdot 6} \alpha$$

$$48\delta - 35\gamma = 0; \quad \delta = \frac{5 \cdot 7}{6 \cdot 8} \gamma = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} \alpha$$

$$80\varepsilon - 63\delta = 0; \quad \varepsilon = \frac{7 \cdot 9}{8 \cdot 10} \delta = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10} \alpha$$

&c.

&c.

Si igitur valor coefficientis primi α constaret, quem quidem jam vidimus esse $= 1$, omnes sequentes coefficientes β, γ, δ , &c. forent cogniti. Verum resolutio alterius æquationis quoque hunc nobis valorem ipsius α patefaciet.

§. XLVII. Substitutis enim seriebus ante traditis in altera æquatione proveniet;

$$\left. \begin{array}{l} 2Ap + 12Bp^3 + 30Cp^5 + 56Dp^7 + 90Ep^9 + \&c. \\ - 2A - 12B - 30C - 56D - \&c. \\ - 2A - 4B - 6C - 8D - 10E - \&c. \\ - 2A - 4B - 6C - 8D - \&c. \\ + 1 + A + B + C + D + \&c. \\ - 4\alpha - 8\beta - 12\gamma - 16\delta - 20\varepsilon - \&c. \\ + 4\alpha + 8\beta + 12\gamma + 16\delta + \&c. \\ + 2\alpha + 2\beta + 2\gamma + 2\delta + 2\varepsilon + \&c. \end{array} \right\} = 0$$

Euleri Opuscula Tom. II.

U

Unde

§. XLVII



Unde simili modo elicitur:

$$2A - 2A + 1 - 2\alpha = 0; \text{ hinc fit } \alpha = \frac{1}{2}$$

$$8B - 3A - 6\beta + 4\alpha = 0; 2.4 B - 1.3 A + 2(2 - \frac{1.3.3}{2.4})\alpha = 0$$

$$24C - 15B - 10\gamma + 8\beta = 0; 4.6 C - 3.5 B + 2(4 - \frac{3.5.5}{4.6})\beta = 0$$

$$48D - 35C - 14\delta + 12\gamma = 0; 6.8 D - 5.7 C + 2(6 - \frac{5.7.7}{6.8})\gamma = 0$$

$$80E - 63D - 18\epsilon + 16\delta = 0; 8.10 E - 7.9 D + 2(8 - \frac{7.9.9}{8.10})\delta = 0$$

§. XLVIII. Cognito igitur valore ipsius $\alpha = \frac{1}{2}$ altera series s , quæ logarithmum ipsius p involvit, tota innotescit, erit enim:

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{1.1.3}{2.2.4}$$

$$\gamma = \frac{1.1.3.3.5}{2.2.4.4.6}$$

$$\delta = \frac{1.1.3.3.5.5.7}{2.2.4.4.6.6.8}$$

$$\epsilon = \frac{1.1.3.3.5.5.7.7.9}{2.2.4.4.6.6.8.8.10}$$

&c.

$$\text{fitque hinc } s = \alpha p p + \beta p^4 + \gamma p^6 + \delta p^8 + \epsilon p^{10} + \&c.$$

§. XLIX. Quod autem ad alteram seriem attinet

$$r = 1 + A p^2 + B p^4 + C p^6 + D p^8 + E p^{10} + \&c.$$

primus



primus coefficientis A hinc manet indeterminatus, cujus rei ratio est, quod has series ex æquatione differentiali secundi gradus eliciimus, quæ duplici determinatione indiget, ut ad nostrum casum accommodetur. Quare valorem hujus coefficientis A ex ipsa curvæ natura definiri oportet, eo autem invento, reliqui innotescent ex his formulis, ad quas superiores redeunt.

$$B = \frac{1.3}{2.4} A - \frac{1}{2} \alpha \left(\frac{3}{2.2} + \frac{1}{1.1} \right)$$

$$C = \frac{3.5}{2.4} B - \frac{1}{6} \beta \left(\frac{3}{3.3} + \frac{1}{2.2} \right)$$

$$D = \frac{5.7}{6.8} C - \frac{1}{2} \gamma \left(\frac{3}{4.4} + \frac{1}{4.4} \right)$$

$$E = \frac{7.9}{8.10} D - \frac{1}{2} \delta \left(\frac{3}{5.5} + \frac{1}{4.4} \right)$$

§. L. His autem omnibus coefficientibus inventis ad datam quamvis abscissam GP = p, valor respondentis applicatæ PQ = q ita definitur, ut sit

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \&c.$$

$$- \alpha p^3 - \beta p^5 - \gamma p^7 - \delta p^9 - \&c.$$

quæ series si abscissa p fuerit unitate multo minor, satis promptè convergit, ut inde valor ipsius q cognosci queat. Hinc vero etiam applicatæ, quæ abscissis multo majoribus unitate respondent,

definiri poterunt, quia abscissæ $\frac{1}{p}$ respondet applicata $\frac{q}{p}$. Quare si abscissa unitate multo major ponatur = P eique respondens applicata = Q ob $p = \frac{1}{P}$ & $q = p Q = \frac{Q}{P}$ erit

$$L 2$$

$$Q =$$

$$Q = P + AP^{-1} + BP^{-2} + CP^{-3} + DP^{-4} + \&c. \\ + \alpha P^{-1} IP + \beta P^{-2} IP + \gamma P^{-3} IP + \delta P^{-4} IP + \&c.$$

Hinc si abscissa P fiat infinita erit

$$Q = P + \frac{\alpha IP}{P} \text{ seu } Q - P = \frac{\alpha IP}{P}$$

unde natura rami Dq in infinitum extensi & ad asytmotam CV appropinquantis colligitur.

§. LI. Quia porro novimus, si $p = 1$ fore $q = \frac{\pi}{2}$ pro hoc casu æquatio inventa hanc formam ob $1 =$ induet

$$\frac{\pi}{2} = 1 + A + B + C + D + E + \&c.$$

Cum igitur valor A nondum sit definitus, reliqui vero B, C, D &c. ab eo pendeant, hæc æquatio conditionem continet, qua valor ipsius A determinatur. Ita scilicet valorem ipsius A comparatum esse oportet, ut summa seriei infinitæ $1 + A + B + C + \&c.$

fiat $\frac{\pi}{2}$. Verum si valores reliquarum litterarum B, C, D &c. qui ab A pendent, evolvantur, tam complicatæ resultant expressiones, ut hinc valor ipsius A neutiquam erui possit.

§. LII. Ad hanc constantem A determinandam alia patet via, si datæ cujuspiam ellipsis perimeter ex altera formula in numeris fuerit inventa. Quæ methodus cum requirat, ut omnes coefficients in fractionibus decimalibus evolvantur, computo peracto reperietur;

α =

$\frac{1}{2}P^{-1} + \&c.$
 $\frac{1}{2}P^{-1} P +$

symptom CV ip

ore $q = \frac{r}{2}$ po

induct

c.

vero B, C, D &c.
 ntinet, qua valor
 us A comparatur
 - B + C + &c

ni B, C, D &c. qui

tant expressiones,

randam alia paret
 formula in nume-
 , ut omnes coeffi-
 computo perado

$\alpha = 0,5000000000$ A quæritur
 $\beta = 0,1675000000$; B = 0,3750000000 A — 0,1093750000
 $\gamma = 0,1171875000$; C = 0,2343750000 A — 0,0820312500
 $\delta = 0,0854492188$; D = 0,1708984375 A — 0,0641886393
 $\epsilon = 0,0672912598$; E = 0,1345825195 A = 0,0524978638
 $\zeta = 0,0555152893$; F = 0,1110305786 A — 0,0443481445
 $\eta = 0,0472540855$; G = 0,0945081711 A — 0,0383663416
 $\theta = 0,0411363691$; H = 0,0822727382 A — 0,0337966962
 $i = 0,0364228268$; I = 0,0728456536 A — 0,0301949487
 $\kappa = 0,0326793696$; K = 0,0653587392 A — 0,0272843726
 &c.

Hisque valoribus inventis, si abscissa sit $CP = p$, valor ap-
 plicatæ q ita definitur ut sit:

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + Gp^{14} + Hp^{16} +$$

$$Ip^{18} + Kp^{20} + \&c.$$

$$- ppIp. (\alpha + \beta p^2 + \gamma p^4 + \delta p^6 + \epsilon p^8 + \zeta p^{10} + \eta p^{12} + \theta p^{14} + i p^{16}$$

$$+ \kappa p^{18} + \&c.)$$

§. LIII. Deinde vero supra ejusdem applicatæ q valorem
 ita invenimus expressum ut sit:

$$q = \frac{r}{2} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} (1 - pp) - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} (1 - pp)^2 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \right.$$

$$\left. (1 - pp)^3 + \&c. \right)$$

Nunc igitur ex utraque formula pro eodem quopiam valo-
 U 3 re



re ipsius p eruamus valorem ipsius q , ut deinceps ex æqualitate horum duorum elicere queamus valorem coefficientis A . Pro p vero non nimis exiguum fractionem substitui conveniet, ne expressio posterior nimis lente convergat, tam parvum tamen assumamus ut coefficientes pro superiore forma computati valori q ad 10 figuras inveniendis sufficiant.

§. LIV. Ponamus ergo ad commodum calculi $p = \frac{2}{5}$. erit in logarithmis hyperbolicis:

$$-lp = 1,60943791243$$

Jam vero est

$$\alpha p p = 0,02000000000$$

$$\beta p^2 = 0,00030000000$$

$$\gamma p^3 = 0,00000750000$$

$$\delta p^4 = 0,00000021875$$

$$\epsilon p^5 = 0,00000000689$$

$$\zeta p^6 = 0,00000000023$$

$$\eta p^7 = 0,00000000001$$

$$0,02030772588$$

coefficientis ipsius $-lp$

$$1,60943791243$$

$$0,03268402394$$

productum.

Deinde est

A_p



ps ex æqualitate
ientis A. Pro p
onveniet, ne ex
rvum tamen affu-
putati valori q ad

caluli $p = \frac{1}{2}$.

$Ap^3 =$	0,04000000000 A	
$Bp^4 =$	0,00060000000 A	— 0,00017500000
$Cp^6 =$	0,00001500000 A	— 0,00000525000
$Dp^8 =$	0,00000043750 A	— 0,00000016432
$Ep^{10} =$	0,00000001378 A	— 0,00000000533
$Fp^{12} =$	0,00000000045 A	— 0,00000000016
$Gp^{14} =$	0,00000000002 A	— 0,00000000001

$$0,04061545175 A \quad - \quad 0,00018041987$$

Ex his conficitur

$$q = 0,04061545175 A + 1,032503250360407$$

§. LV. Nunc eundem valorem ipsius q ex altera æquatione quæramus, & cum sit $p = \frac{1}{2}$, erit $1 - pp = \frac{24}{25}$ sit $nn = \frac{24}{25}$ erit

$$q = \frac{\pi}{1} \left(1 - \frac{1 \cdot 1}{2 \cdot 2} nn - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} n^3 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^5 - \&c. \right)$$

ponatur ad abbreviandum

$$q = \frac{\pi}{2} - \mathcal{A}n^2 - \mathcal{B}n^4 - \mathcal{C}n^6 - \mathcal{D}n^8 - \mathcal{E}n^{10} - \&c.$$

us — p

Verum hoc casu ob $nn = \frac{24}{25}$ series ista nimis lente convergit, quam ut hinc valor ipsius q satis exacte elici queat, quare ut utrinque parem convergentiam obtineamus ponamus $p = \frac{1}{\sqrt{2}}$, ut sit tam $pp = \frac{1}{2}$ quam $nn = \frac{1}{2}$; calculum vero tantum ad 6 figuras expediamus: eritque

Ap

APP



$Ap^0 = 0, 500000 A$	
$Bp^4 = 0, 093750 A - 0, 027344$	
$Cp^6 = 0, 019297 A - 0, 010254$	
$Dp^8 = 0, 010681 A - 0, 004012$	
$Ep^{10} = 0, 004206 A - 0, 001640$	
$Fp^{12} = 0, 001735 A - 0, 000693$	
$Gp^{14} = 0, 000738 A - 0, 000300$	
$Hp^{16} = 0, 000321 A - 0, 000132$	
$Ip^{18} = 0, 000142 A - 0, 000059$	
$Kp^{20} = 0, 000064 A - 0, 000026$	
Summa reliq: 60 A —	24

$$\text{Sa. om. } 0, 640994 A - 0, 044484 + 0, 310497. \quad \frac{1}{p} + 1$$

ergo $q = 1, 066592 + 0, 640994 A$
 at altera expressio dat $q = 1, 350647$, unde fit

$$A = \frac{284055}{640994} = 0, 443147$$

§. LVI. Quamquam hic valor non ultra 6 figuras extenditur, tamen casui non tribuendum videtur, quod iste numerus inventus 0, 443147 a logarithmo binarii 0, 69314718 unitatis quadrante 0, 25 præcise deficiat. Quæ conjectura si veritati esset consentanea, valorem li. $\frac{1}{2}$ e A ad plurimas figuras exhibere liceret, cum enim sit

$$12 = 0, 6931471805599453094172321$$

foret $A = 12 - \frac{1}{2}$ ideoque

$$A = 0, 4431471805599453094172321.$$

Quod

Quod autem valor coefficientis hujus A sit revera $= 12 - \frac{1}{4}$, sequenti modo demonstro, hancque conjecturam confirmo.

§. LVII. Comparo scilicet arcum ellipticum AYP, cujus Fig. 2. semiaxes $AC = 1$, $CP = p$ cum arcu parabolico AZS super eodem axe AC descripto, qui in A cum ellipsi communem habeat curvaturam. Sumta abscissa communi $AX = x$, sit applicata ellipsis $XY = y$ & parabolæ $XZ = z$, erit $y = p \sqrt{(2x - xx)}$ & $z = p \sqrt{2x}$, ideoque $dy = \frac{p dx (1-x)}{\sqrt{(2x-xx)}}$ & $dz = \frac{p dx}{\sqrt{2x}}$: unde fit

$$\text{arcus ellipticus AY} = \int dx \sqrt{(1 + \frac{pp(1-x)^2}{2x-xx})}$$

$$\text{arcus parabolicus AZ} = \int dx \sqrt{(1 + \frac{pp}{2x})}. \text{ Constat autem}$$

$$\text{effe } AZ = x \sqrt{(1 + \frac{pp}{2x})} + \frac{1}{2} pp \int \frac{\sqrt{(1 + \frac{pp}{2x})} + 1}{\sqrt{(1 + \frac{pp}{2x})} - 1} dx; \text{ Hinc si ponatur}$$

$$x = 1, \text{ erit arcus parabolicus AZS} = \sqrt{(1 + \frac{1}{2} pp)} + \frac{1}{2} pp.$$

$$\frac{\sqrt{(1 + \frac{1}{2} pp)} + 1}{\sqrt{(1 + \frac{1}{2} pp)} - 1}$$

At in formulis integralibus erit:

$$\sqrt{(1 + \frac{pp(1-x)^2}{2x-xx})} = \sqrt{(1 + \frac{pp}{2x} - \frac{pp(3-2x)}{4-2x})}$$

Quia autem comparationem non ad majores ipsius p potestates extendere opus est quam ad secundam: coefficientes enim altiorum ipsius p potestatum ex minoribus jam definivimus, relictis terminis, qui continent p^4 & altiores potestates, erit

Euleri Opuscula Tom. II.

X

$\sqrt{(1$

$$V(1 + \frac{pp(1-x)^2}{2x-xx}) = V(1 + \frac{pp}{2x} - \frac{pp(3-2x)}{2(2-x)}) \text{ ideoque}$$

$$AY = \int dx V(1 + \frac{pp}{2x}) - \frac{1}{2} p p f \frac{dx(3-2x)}{2-x}, \text{ integralibusque}$$

æstu sumtis

$$AY = x V(1 + \frac{pp}{2x}) + \frac{1}{2} p p f \frac{V(1 + \frac{pp}{2x}) + 1}{V(1 + \frac{pp}{2x}) - 1} - \frac{1}{2} p p x - \frac{1}{2} p p f \frac{2-x}{2}$$

Ponatur jam $x = 1$, ut prodeat arcus $AYP = q$, erit

$$q = V(1 + \frac{1}{2} pp) + \frac{1}{2} p p f (V(1 + \frac{1}{2} pp) + 1) - \frac{1}{2} p p f (V(1 + \frac{1}{2} pp) - 1) - \frac{1}{2} pp + \frac{1}{2} p p f 2.$$

§. LVIII. Jam quoniam ad altiores ipsius p potestates

non respicimus, erit $V(1 + \frac{1}{2} pp) = 1 + \frac{1}{4} pp$, unde fiet

$$q = 1 + \frac{1}{4} pp + \frac{1}{4} p p f (2 + \frac{1}{4} pp) - \frac{1}{4} pp f \frac{1}{4} pp - \frac{1}{2} pp + \frac{1}{4} p p f 2$$

ubi pro $f(2 + \frac{1}{4} pp) = f 2 + \frac{1}{8} pp$ scribere licet $f 2$, ita ut

$$\text{fit } q = 1 - \frac{1}{4} pp + \frac{1}{2} p p f 2 - \frac{1}{2} p p f p + \frac{1}{2} p p f 2$$

$$\text{seu } q = 1 - \frac{1}{2} p p f p + p p (f 2 - \frac{1}{4})$$

unde perspicitur coefficientem ipsius pp , quem ante litera A indica-
vimus

vimus esse $= 12 - \frac{1}{4}$, omnino uti ex casu ante computato conjectura sumus consecuti.

§. LIX. Pro curva igitur initio proposita AQDq, si su- Fig. 1.
crit abscissa CP = p & applicata PQ = q , erit

$$q = 1 + A p p + B p^3 + C p^5 + D p^7 + E p^9 + \&c.$$

$$-(\alpha p p + \beta p^3 + \gamma p^5 + \delta p^7 + \epsilon p^9 + \&c.) \text{ } l p$$

ubi coefficientes ita determinantur:

$$\begin{array}{l|l} A = 12 - \frac{1}{4} & \alpha = \frac{1}{2} \\ B = \frac{1.3}{2.4} A - \frac{1}{2}(\alpha - \beta) + \frac{1}{2} \cdot \frac{6}{2} & \beta = \frac{1.3}{2.4} \alpha \\ C = \frac{3.5}{4.6} B - \frac{1}{3}(\beta - \gamma) + \frac{1}{4} \cdot \frac{7}{3} & \gamma = \frac{3.5}{4.6} \beta \\ D = \frac{5.7}{6.8} C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6} \cdot \frac{8}{4} & \delta = \frac{5.7}{6.8} \gamma \\ E = \frac{7.9}{8.10} D - \frac{1}{5}(\delta - \epsilon) + \frac{1}{8} \cdot \frac{9}{5} & \epsilon = \frac{7.9}{8.10} \delta \\ F = \frac{9.11}{10.12} E - \frac{1}{6}(\epsilon - \zeta) + \frac{1}{10} \cdot \frac{11}{6} & \zeta = \frac{9.11}{10.12} \epsilon \\ & \&c. \end{array}$$

series hæc valde convergit, si abscissa p fuerit fractio valde parva,
sin autem sit unitate multo major, iisdem manentibus coefficienti-
bus erit

X 2

q =



$$q = p + \frac{A}{p} + \frac{B}{p^3} + \frac{C}{p^5} + \frac{D}{p^7} + \frac{E}{p^9} + \&c.$$

$$+ \left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\epsilon}{p^9} + \&c. \right) fp$$

§. LX. Verum si abscissa p non multum ab unitate discrepet, uti conveniet hac serie supra §. XXVI. inventa

$$q = 1 + fp \left\{ \begin{aligned} & \left(\frac{\pi}{2} - 1 \right) + \left(\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} - 1 \right) (1 - fp) + \\ & \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4} \cdot \frac{\pi}{2} - 1 \right) (1 - fp)^2 + \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 6} \right. \\ & \left. \frac{\pi}{2} - 1 \right) (1 - fp)^3 + \&c. \end{aligned} \right.$$

quæ etiam ex natura ellipsis in hanc convertitur

$$q = p + \frac{1}{p} \left\{ \begin{aligned} & \left(\frac{\pi}{2} - 1 \right) - \left(\frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{\pi}{2} - 1 \right) \frac{(1 - pp)}{fp} + \\ & \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4} \cdot \frac{\pi}{2} - 1 \right) \frac{(1 - pp)^2}{p^2} - \left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 2 \cdot 4 \cdot 6} \right. \\ & \left. \frac{\pi}{2} - 1 \right) \frac{(1 - pp)^3}{p^3} + \&c. \end{aligned} \right.$$

unde prout fuerit vel $p > 1$ vel $p < 1$ eam eligere licet, cujus termini vel iisdem signis procedant, vel alternantibus. Plerumque autem præstat ad summam proxime definiendam signa eligere alternantia.

Pro-

Problema.

§. LXI. Datis axibus conjugatis ellipsis, in numeris proxime exhibere ejus perimetrum.

Solutio.

Sint semiaxes ellipsis 1 & p , & quadrans perimetri = q , atque per formulas inventas valor ipsius q in numeris definiri poterit, dummodo ea eligatur, cujus termini maxime convergant. Quatuor autem adepti sumus formulas quæ sunt:

$$\text{I. } q = 1 + App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} \text{ \&c.}$$

$$- (app + \beta p^4 + \gamma p^6 + \delta p^8 + \epsilon p^{10} + \zeta p^{12} \text{ \&c.}) \text{ } tp$$

$$\text{II. } q = p + A \frac{1}{p} + B \frac{1}{p^3} + C \frac{1}{p^5} + D \frac{1}{p^7} + E \frac{1}{p^9} + F \frac{1}{p^{11}} + \text{\&c.}$$

$$+ \left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\epsilon}{p^9} + \frac{\zeta}{p^{11}} + \text{\&c.} \right) tp$$

$$\text{III. } q = 1 + pp (\mathcal{A} + \mathcal{B} (1 - pp) + \mathcal{C} (1 - pp)^2 + \mathcal{D} (1 - pp)^3 + \mathcal{E} (1 - pp)^4 + \text{\&c.})$$

$$\text{IV. } q = p + \frac{1}{p} (\mathcal{A} - \mathcal{B} \frac{(1 - pp)}{pp} + \mathcal{C} \frac{(1 - pp)^2}{p^2} - \mathcal{D} \frac{(1 - pp)^3}{p^3} + \mathcal{E} \frac{(1 - pp)^4}{p^4} - \text{\&c.})$$

Horum



642941



Horum autem tergeminarum coefficientium valores sunt in numeris:

A = 0,44314718056	$\alpha = 0,50000000000$	$\Psi = 0,57079632679$
B = 0,05680519271	$\beta = 0,18750000000$	$\mathfrak{B} = 0,17809724510$
C = 0,02183137044	$\gamma = 0,11718750000$	$\mathfrak{C} = 0,10446616728$
D = 0,01154452143	$\delta = 0,08544921875$	$\mathfrak{D} = 0,07378655152$
E = 0,00714200029	$\epsilon = 0,06729125977$	$\mathfrak{E} = 0,05700863665$
F = 0,00485474337	$\zeta = 0,05551527931$	$\mathfrak{F} = 0,04643855029$
G = 0,00351468795	$\eta = 0,04725408554$	$\mathfrak{G} = 0,03917161591$
H = 0,00266223578	$\theta = 0,04113636911$	$\mathfrak{H} = 0,03386971991$
I = 0,00208639732	$i = 0,03642282682$	$\mathfrak{I} = 0,02983116632$
K = 0,00167916842	$\kappa = 0,03267936962$	$\mathfrak{K} = 0,02665267507$
		$\mathfrak{J} = 0,02408604339$

Hinc pro quavis ellipsis specie habebitur series convergens, unde ejus perimeter definiri poterit, veluti

$$\text{si ponatur } p = \frac{1}{10} \text{ erit} \quad q = 1,015993545021$$

$$\text{si sit } p = \frac{1}{5} \text{ erit} \quad q = 1,05050222709$$

$$\text{si sit } p = \frac{1}{\sqrt{2}} \text{ erit} \quad q = 1,3506429$$



rium values are

20,570,963,263

20,178,097,24510

0,104,466,16728

0,073,786,3132

0,057,008,63665

1,046,438,31029

1,039,171,61,91

0,338,637,1991

02983116638

1266,826,7107

2408604339

convergence,

f



